Title:	A Simple Derivation of the Fractional-N Phase Noise Result
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1 INTRODUCTION

A p^{th}-order MASH fractional-N phase-locked loop is considered in the discussions that follow. As we will see, the colored fractional-N phase noise portion (less any additional filtering by the loop filter) as seen on a spectrum analyzer at the output of the PLL is given by

$$\mathscr{L}(f) = 10\log_{10}\left\{ \frac{(2\pi)^2}{12F_{PD}} \left[2\sin\left(\frac{\pi f}{F_{PD}} \right) \right]^{2(p-1)} \right\} dBc/Hz \tag{1}$$

This result can be traced back at least as far as Miller's fine patent on the MASH architecture [1]. A simple derivation of (1) follows.

2 P^{TH}-ORDER Δ-Σ FRACTIONAL-N PLL NOISE

The Δ-Σ MASH architecture[1] is such that all of the internal quantization noise terms cancel except for one, resulting in the z-transform of the total noise contribution being given by

$$n(z) = \left(1 - z^{-1}\right)^p e_q(z) \tag{2}$$

for a p^{th}-order MASH, and $e_q(z)$ is the z-transform of the ideally random quantization noise that is assumed to be uniformly distributed over the range [–1/2,1/2]. In this present context, for example, if the PLL's feedback divider were set to divide by 133 for a specific reference time period whereas the desired PLL output frequency is 133.245 × F_{PD}, e_q for that time interval would be 0.245. The z-transform of the corresponding radian frequency error is then given by

$$\omega_q(z) = \frac{2\pi}{T_{PD}} \left(1 - z^{-1}\right)^p e_q(z) \tag{3}$$

where T_{PD} is the time-period at the phase detector. In order to express the error in terms of phase at the PLL output, the radian frequency (3) must be integrated which corresponds to a multiplication by $T_{PD} / (1 - z^{-1})$ resulting in

$$\begin{aligned} \theta_e(z) &= \frac{2\pi}{T_{PD}} \left(1 - z^{-1}\right)^p e_q(z) \frac{T_{PD}}{1 - z^{-1}} \\ &= 2\pi e_q(z) \left(1 - z^{-1}\right)^{p-1} \end{aligned} \tag{4}$$

Assuming that $e_q(.)$ is uniformly distributed on the span [–1/2,1/2], the one-sided power spectral density of θ_e is given by

[1] Chapter 8 of [2].

$$S_\theta(f) = \frac{(2\pi)^2}{6F_{PD}} \left|1 - z^{-1}\right|^{2(p-1)} \quad rad^2 / Hz \tag{5}$$

where z = exp(j 2 π f T_{PD}), and F_{PD} = 1 / T_{PD}. The valid frequency range for (5) is $0 \le f \le F_{PD}/2$.

In order to convert (5) into the phase noise level that will be observed on a spectrum analyzer at the PLL's output, we can make use of the approximation that

$$\mathscr{L}(f) = 20\log_{10}\left(\frac{\Delta\theta}{2}\right) \quad dBc / Hz \tag{6}$$

where Δθ is the *peak* phase noise observed in a 1 Hz bandwidth at a frequency offset from DC of f. Taking the Δθ to be equal to $\theta_{RMS}\sqrt{2}$, we can use (5) to write (6) as

$$\begin{aligned}\mathscr{L}(f) &= 10\log_{10}\left[\frac{(\Delta\theta)^2}{4}\right] = 10\log_{10}\left[\frac{S_\theta(f)}{2}\right] \\ &= 10\log_{10}\left[\frac{(2\pi)^2}{12F_{PD}}\left|1 - z^{-1}\right|^{2(p-1)}\right]\end{aligned} \tag{7}$$

Substituting z = cos(2π f / F_{PD}) + j sin(2π f / F_{PD}) into (7) and collecting terms leads to the final result.

3 REFERENCES

1. Miller, B.M., "Multiple-Modulator Fractional-N Divider," U.S. Patent 5,038,117, filed Sept. 7, 1990, issued Aug. 6, 1991.
2. Crawford, J.A., *Advanced Phase-Lock Techniques*, Artech House, 2008.
3. ____________, *Frequency Synthesizer Design Handbook*, Artech House, 1994.