Approximation for $\sqrt{I^2+Q^2}$

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A commonly used approximation for this quantity is

$$\sqrt{I^2+Q^2} \approx \max\left(|I|,|Q|\right) + 0.375\min\left(|I|,|Q|\right) \tag{1}$$

Assuming a large number of OFDM subcarriers in the intended application, I and Q appear to both be Gaussian thereby making r a Rayleigh-distributed random variable where

$$r = \sqrt{I^2+Q^2} \tag{2}$$

We are interested in the ratio

$$\eta = \frac{\max\left(|I|,|Q|\right)+0.375\min\left(|I|,|Q|\right)}{\sqrt{I^2+Q^2}} \tag{3}$$

From symmetry, it suffices to consider a limited range for θ as $0 \le \theta \le \pi/4$. Letting

$$\begin{aligned} I &= r_o\cos(\theta) \\ Q &= r_o\sin(\theta) \end{aligned} \tag{4}$$

leads to

$$\begin{aligned} \max\left(|I|,|Q|\right) &= r_o\cos(\theta) \\ \min\left(|I|,|Q|\right) &= r_o\sin(\theta) \end{aligned} \tag{5}$$

and

$$r \approx r_o\left[\cos(\theta)+\gamma\sin(\theta)\right] \tag{6}$$

where $\gamma = 0.375$. It is worthwhile evaluating just how good this choice for γ really is.

We seek then to minimize the mean-square error given by

$$Error = \int_0^{\pi/4}\left[\cos(\theta)+\gamma\sin(\theta)-1\right]^2 d\theta \tag{7}$$

Minimizing this error with respect to γ by differentiation leads to the equation

$$\int_0^{\pi/4} 2\cos(\theta)\sin(\theta)+2\gamma\sin^2(\theta)-2\sin(\theta)\,d\theta = 0 \tag{8}$$

which upon solving gives the optimal value for γ as

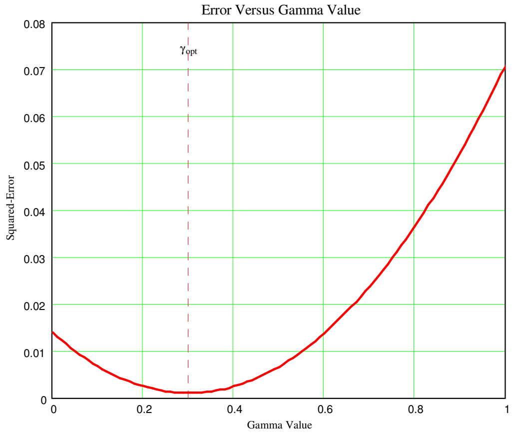
$$\gamma = \frac{\frac{1}{2}+2\left(\frac{\sqrt{2}}{2}-1\right)}{-2\left(\frac{\pi}{8}-\frac{1}{4}\right)} \approx 0.3006 \tag{9}$$

Since $0.3 \approx \frac{1}{2}-\frac{1}{8}-\frac{1}{16} = 0.3125$, it's attractive to consider using this binary-weighted value for γ but with the subtractions involved, this is not as convenient for a digital solution as desired. We note that

$$\frac{Error(\gamma=0.375)}{Error(\gamma=0.3125)} = 1.762 \tag{10}$$

Therefore, the improved value for γ of 0.3125 is helpful by almost a factor of two, but its implementation in an ASIC is still not as attractive as a value of 0.375= ¼ + 1/8.

Figure 1 Error Versus Gamma Value Choice[1]



The worst-case peak-to-peak error can be kept to within ±0.50 dB by choosing $\gamma = 0.34375 = (1/4+1/16+1/32)$ but the complexity involved in this computation is more than desired.

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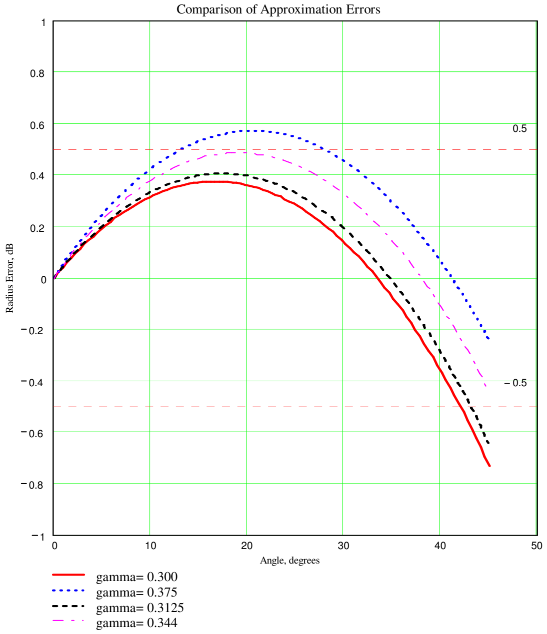
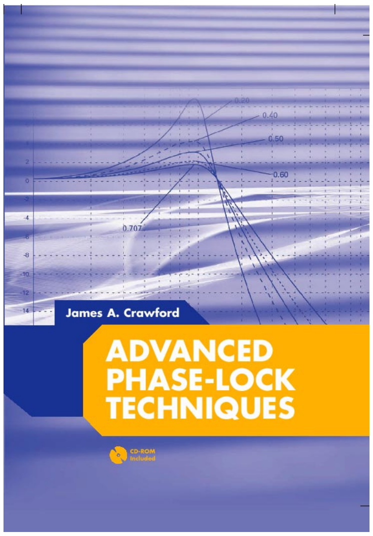
[1] From M11589

Figure 2 Peak-to-Peak Errors Versus Gamma Choice



In the end, $\gamma = 0.375 = \frac{1}{4} + \frac{1}{8}$ was adopted for its ease of implementation and its comparatively good error performance.



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Chapter	Brief Description	*Pages*
1	*Phase-Locked Systems—A High-Level Perspective* An expansive, multi-disciplined view of the PLL, its history, and its wide application.	26
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3	*Fundamental Limits* A detailed discussion of the many fundamental limits that PLL designers may have to be attentive to or else never achieve their lofty performance objectives, e.g., Paley-Wiener Criterion, Poisson Sum, Time-Bandwidth Product.	38
4	*Noise in PLL-Based Systems* An extensive look at noise, its sources, and its modeling in PLL systems. Includes special attention to 1/*f* noise, and the creation of custom noise sources that exhibit specific power spectral densities.	66
5	*System Performance* A detailed look at phase noise and clock-jitter, and their effects on system performance. Attention given to transmitters, receivers, and specific signaling waveforms like OFDM, M-QAM, M-PSK. Relationships between EVM and image suppression are presented for the first time. The effect of phase noise on channel capacity and channel cutoff rate are also developed.	48
6	*Fundamental Concepts for Continuous-Time Systems* A thorough examination of the classical continuous-time PLL up through 4th-order. The powerful Haggai constant phase-margin architecture is presented along with the type-3 PLL. Pseudo-continuous PLL systems (the most common PLL type in use today) are examined rigorously. Transient response calculation methods, 9 in total, are discussed in detail.	71
7	*Fundamental Concepts for Sampled-Data Control Systems* A thorough discussion of sampling effects in continuous-time systems is developed in terms of the *z*-transform, and closed-form results given through 4th-order.	32
8	*Fractional-N Frequency Synthesizers* A historic look at the fractional-N frequency synthesis method based on the U.S. patent record is first presented, followed by a thorough treatment of the concept based on Δ-Σ methods.	54
9	*Oscillators* An exhaustive look at oscillator fundamentals, configurations, and their use in PLL systems.	62
10	*Clock and Data Recovery* Bit synchronization and clock recovery are developed in rigorous terms and compared to the theoretical performance attainable as dictated by the Cramer-Rao bound.	52