Substantiation of the Water Filling Theorem Using Lagrange Multipliers

Shannon's capacity theorem states that in the case of N parallel statistically independent Gaussian channels that the channel capacity C is given by

(1) $$C = \sum_{n=1}^{N} B \log_2 \left(1 + \frac{E_n}{\sigma_n^2} \right)$$

where B is the bandwidth per channel in Hz, and E_n and σ_n^2 are the energy per symbol and noise variance per symbol respectively. We want to find the best allocation of transmit power in order to maximize C under the maximum power constraint that

(2) $$\sum_{n=1}^{N} E_n = E_T$$

The Lagrange Multiplier solution method is given by the following rule[1]:

In order to determine the extreme values of a continuously differentiable function $f(x_1, x_2, \ldots, x_n)$ whose variables are subjected to m continuously differentiable constraining relations given as

(3) $$\varphi_i(x_1, x_2, \ldots x_n) = 0 \quad for \quad i = 1, 2, \ldots, m$$

form the function

(4) $$F = f + \sum_{i=1}^{m} \lambda_i \varphi_i$$

and determine the parameters λ_i and the values of x_k from the n equations

(5) $$\frac{\partial F}{\partial x_k} = 0 \quad for \quad k = 1, 2, \ldots, n$$

and the m equations given by (3).

The objective function that we wish to find the extremum of is then given by

(6) $$\Lambda = \sum_{n=1}^{N} B \log_2 \left(1 + \frac{E_n}{\sigma_n^2} \right) + \lambda \left(E_T - \sum_{n=1}^{N} E_n \right)$$

Differentiating (6) with respect to E_n results in

[1] I.S., Sokolnikoff, R.M. Redheffer, ***Mathematics of Physics and Modern Engineering***, McGraw-Hill Book, 1966

(7) $$\frac{\partial \Lambda}{\partial E_n} = \frac{B}{\ln(2)} \frac{1}{1+\frac{E_n}{\sigma_n^2}} \frac{1}{\sigma_n^2} - \lambda = 0$$

from which we obtain the requirement that

(8) $$\frac{B}{\ln(2)} \frac{1}{1+\frac{E_n}{\sigma_n^2}} \frac{1}{\sigma_n^2} = \lambda'$$

for any (positive) choice of constant λ' for all n which means that $E_n + \sigma_n^2 = \mu$ a constant for all n thereby providing us the classical "water filling" criteria for maximizing the system capacity. Since it is not possible to have a negative value for E_n, the minimum allowable value for μ is the maximum value of the σ_n^2. Substituting this result into the capacity formula, we find that

(9) $$C = \sum_{n=1}^{N} B \log_2\left(1+\frac{\mu-\sigma_n^2}{\sigma_n^2}\right) = \sum_{n=1}^{N} B \log_2\left(\frac{\mu}{\sigma_n^2}\right)$$

Parallel Gaussian BPSK Channels

Now assume that we have N parallel Gaussian channels each supporting uncoded BPSK, and we wish to minimize the overall average bit error rate (BER) subject to the same total power constraint as used above. The average BER is given by

(10) $$BER = \frac{1}{N}\sum_{n=1}^{N}\frac{1}{2} erfc\left(\sqrt{\frac{E_n}{\sigma_n^2}}\right)$$

and the total power constraint is given again by

(11) $$\sum_{n=1}^{N} E_n = E_T$$

Employing the Lagrange multiplier method, the objective function that we seek the extremum values for is given by

(12) $$\Lambda = \frac{1}{N}\sum_{n=1}^{N}\frac{1}{2} erfc\left(\sqrt{\frac{E_n}{\sigma_n^2}}\right) + \lambda\left[E_T - \sum_{n=1}^{N} E_n\right]$$

Differentiating (12) with respect to E_n and setting the derivatives to zero results in the criteria that

(13) $$\lambda = \frac{\exp\left(-\frac{E_n}{\sigma_n^2}\right)}{2N\sigma_n\sqrt{\pi E_n}}$$

which is obviously a transcendental equation in E_n which must be satisfied for all n given a constant parameter λ. This equation can be rearranged to given E_n in terms of the other parameters as

(14) $$E_n + \frac{\sigma_n^2}{2}\ln\left(E_n\right) = -\sigma_n^2 \ln\left[2\lambda N\sigma_n\sqrt{\pi}\right]$$

A non-transcendental result can be obtained if we use the Chernoff bound expression for BER rather than the exact form. In this case, the objective function involved is given by

(15) $$\Lambda = \frac{1}{2N}\sum_{n=1}^{N}\exp\left(-\frac{E_n}{\sigma_n^2}\right) + \lambda\left(\sum_{n=1}^{N}E_n - E_T\right)$$

Upon taking derivatives of (15) and setting them to zero, the relationship for extremum values that results is given by

(16) $$E_n = -\sigma_n^2 \ln\left(2N\sigma_n^2\lambda\right)$$

It is interesting to compare the results for E_n using the exact result found by solving (13) and the non-transcendental result provided by (16). An example case is shown in the Mathcad worksheet that follows[2].

[2] U10333 Minimize Ave BER on Parallel Faded Channels.mcd

Minimizing BER for Parallel Gaussian BPSK Channels

$N := 15$ Number of parallel channels

$nn := 0 .. N-1$

$var0 := (2.\ \ 1.0\ \ 0.75\ \ 0.6\ \ 0.3\ \ 0.1\ \ 0.25\ \ 0.25\ \ 0.25\ \ 2.0\ \ 3.0\ \ 5.0\ \ 1.0\ \ 0.5\ \ 0.03)^T$

$$var_T := \sum_{nn} var0_{nn}$$

$$var_{nn} := \frac{var0_{nn}}{var_T} \cdot N$$

$En := 0.3$

Given

$$\left(\lambda - \frac{e^{\frac{-En}{var}}}{2 \cdot N \cdot \sqrt{var \cdot \pi \cdot En}} \right) \cdot 10^5 = 0$$

$CompEn(\lambda, var) := Find(En)$ Solution for Individual channel energy using rigorous derivation from Lagrange multiplier

$CompEn(0.5, 0.1) = 0.011$

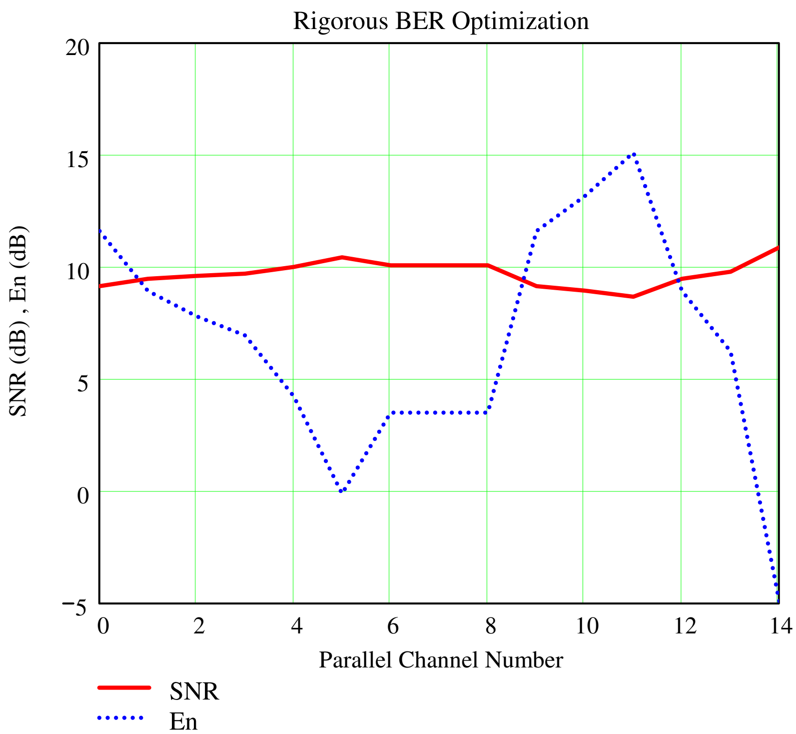
$\lambda := 0.000001$

$En_{nn} := \text{CompEn}\left(\lambda, var_{nn}\right)$ $\sum_{nn} En_{nn} = 123.636$ $AvSNR_{dB} := 10 \cdot \log\left(\frac{1}{N} \cdot \sum_{nn} \frac{En_{nn}}{var_{nn}}\right)$

$AvSNR_{dB} = 9.742$

$BER := \left(\sum_{nn} \frac{1}{2} \cdot \text{erfc}\left(\sqrt{\frac{En_{nn}}{var_{nn}}}\right)\right)$ $BER = 2.129 \times 10^{-4}$

$\sum_{nn} En_{nn} = 123.636$ $10 \cdot \log\left(\frac{\sum_{nn} En_{nn}}{\sum_{nn} var_{nn}}\right) = 9.161$



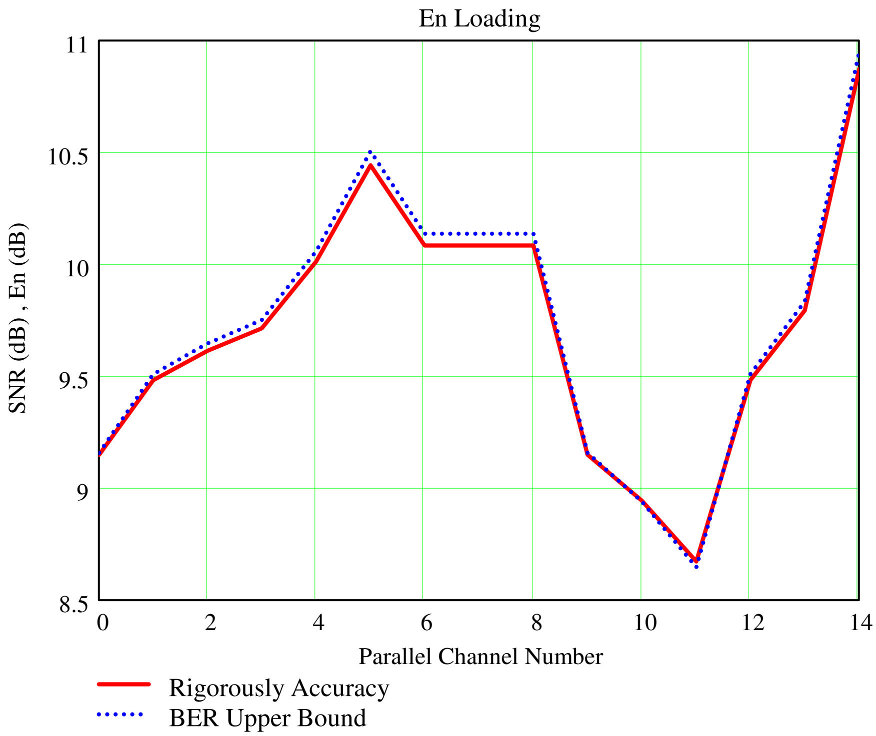
$\lambda 2 := 0.00001$

$En2_{nn} := -var_{nn} \cdot \ln\left(\lambda 2 \cdot N \cdot var_{nn}\right)$

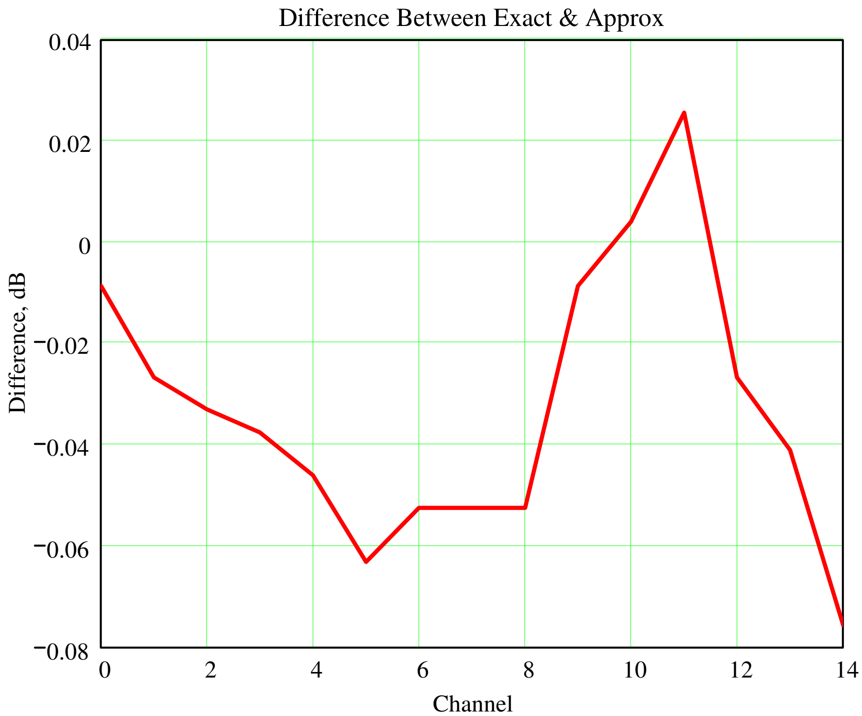
$En2^T =$

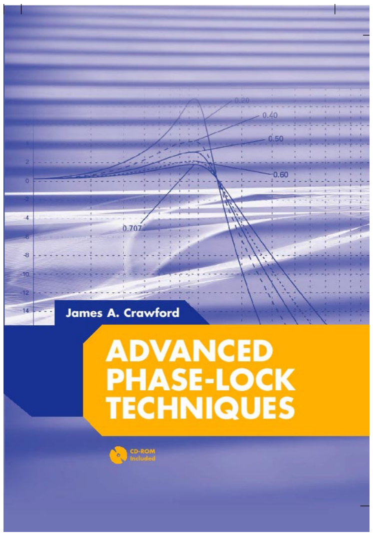
	0	1	2	3	4	5	6	7	8	9
0	14.513	7.867	6.09	4.99	2.678	0.99	2.272	2.272	2.272	14.513

$\sum_{nn} \left(En2_{nn}\right)^2 = 2.117 \times 10^3$ $10 \cdot \log\left(\frac{\sum_{nn} En2_{nn}}{\sum_{nn} var_{nn}}\right) = 9.168$



$$del_{nn} := 10\cdot\log\left(\frac{En_{nn}}{var_{nn}}\right) - 10\cdot\log\left(\frac{En2_{nn}}{var_{nn}}\right)$$





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Chapter	**Brief Description**	***Pages***
1	*Phase-Locked Systems—A High-Level Perspective* An expansive, multi-disciplined view of the PLL, its history, and its wide application.	26
2	*Design Notes* A compilation of design notes and formulas that are developed in details separately in the text. Includes an exhaustive list of closed-form results for the classic type-2 PLL, many of which have not been published before.	44
3	*Fundamental Limits* A detailed discussion of the many fundamental limits that PLL designers may have to be attentive to or else never achieve their lofty performance objectives, e.g., Paley-Wiener Criterion, Poisson Sum, Time-Bandwidth Product.	38
4	*Noise in PLL-Based Systems* An extensive look at noise, its sources, and its modeling in PLL systems. Includes special attention to $1/f$ noise, and the creation of custom noise sources that exhibit specific power spectral densities.	66
5	*System Performance* A detailed look at phase noise and clock-jitter, and their effects on system performance. Attention given to transmitters, receivers, and specific signaling waveforms like OFDM, M-QAM, M-PSK. Relationships between EVM and image suppression are presented for the first time. The effect of phase noise on channel capacity and channel cutoff rate are also developed.	48
6	*Fundamental Concepts for Continuous-Time Systems* A thorough examination of the classical continuous-time PLL up through 4^{th}-order. The powerful Haggai constant phase-margin architecture is presented along with the type-3 PLL. Pseudo-continuous PLL systems (the most common PLL type in use today) are examined rigorously. Transient response calculation methods, 9 in total, are discussed in detail.	71
7	*Fundamental Concepts for Sampled-Data Control Systems* A thorough discussion of sampling effects in continuous-time systems is developed in terms of the z-transform, and closed-form results given through 4^{th}-order.	32
8	*Fractional-N Frequency Synthesizers* A historic look at the fractional-N frequency synthesis method based on the U.S. patent record is first presented, followed by a thorough treatment of the concept based on Δ-Σ methods.	54
9	*Oscillators* An exhaustive look at oscillator fundamentals, configurations, and their use in PLL systems.	62
10	*Clock and Data Recovery* Bit synchronization and clock recovery are developed in rigorous terms and compared to the theoretical performance attainable as dictated by the Cramer-Rao bound.	52