One

Perfecting My Swing- Part 1

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Synopsis

I have had a love affair with precision clocks as long as I can remember. More specifically, Huygens-style pendulum clocks. The mathematics of its motion have always been interesting to me, and when I learned the precise period of the pendulum's swing is related to elliptic integrals, I was totally hooked.

Consequently last year, in the midst of my father's down-sizing activities, his grandfather clock (which my brother and I gave to our parents for their 25th wedding anniversary) needed a new home. The search for a new home was very short-lived; it now resides in my living room.

Sometime in high school, I stumbled on to a book which described an electrical/electronic means to improve a pendulum clock's precision more or less without limit by slaving the pendulum's swing to a precision quartz oscillator. A recent search on the IEEE website and a search of the US patent archives turned up many such prior efforts by others. But I decided to put my own spin on the topic using modern-day components.

In short, this article describes the approach I have taken using (i) an Arduino Mega microprocessor, (ii) a precision GPS receiver with 1 PPS output, and (iii) some simple driver electronics to effectively make my dad's grandfather clock keep perfect time. I still have to lift the counter-weights back into place every 5-7 days to provide power to the clock mechanism, but the time-keeping should now be perfect. With this add-on to my grandfather clock, the pendulum's swing has been made perfect.

1 Pendulum Clock History

The Huygens pendulum clock was invented by[1] Christiaan Huygens in 1656, and was considered the most precise timekeeping device available until the 1930's. The Wikipedia article is fairly short and well worth the time it takes to read it.

The history behind the first accurate maritime chronometer was made into a miniseries in the year 2000. The difficulties associated with a ship at sea made the pendulum clock unworkable, but having a precise time keeping source was crucial for calculating the longitude of a ship at sea. Recognition of this by the British Parliament led to the establishment of a substantial reward[2] in 1714 for anyone who could solve the problem. The journey to solve the longitude accuracy problem spanned more than 50 years! Back now to the classic Huygens pendulum as shown in Figure 1.

Development of the differential equation describing the pendulum bob's motion can be found in [1], along with the numerical methods to precisely compute elliptic integrals of the first kind. I will not revisit these topics here for sake of brevity.

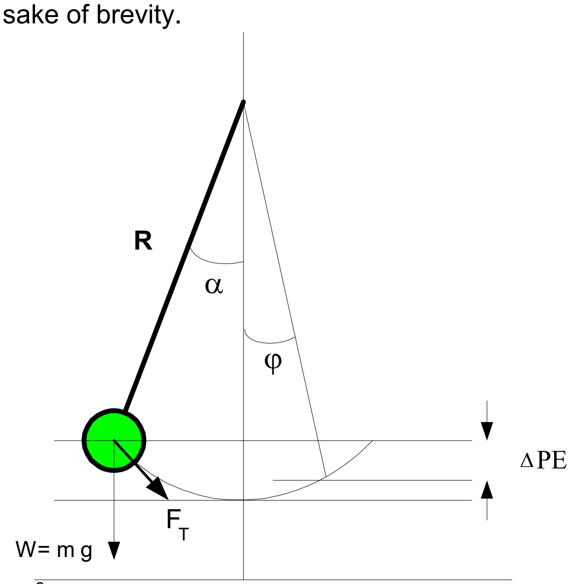

Figure 1 Classical pendulum[3] in which the weight of the pendulum-bob is given by *m*, the maximum angular swing by α, the instantaneous angular excursion from vertical by φ, and the pendulum's length by *R*. The change in potential energy is denoted by ΔPE and the pendulum's arm is assumed to be weightless.

1 Basic Concepts

It so happens Huygens is also credited with discovering the nature of *phase-locking* between pendulum clocks in the form of *injection locking*. A summary of his findings can be found in §1.1.1 of [2]. The method used to synchronize the pendulum's swing with a GPS-based 1 PPS source is best described as injection locking which is the subject of Adler's paper [3] from 1946. This topic is looked at shortly in §1.2.

1 From Wikipedia article "Pendulum clock".
2 Known as the *Longitude Act*.
3 Figure 1 from [1].

As developed in [1], the period of the pendulum's oscillatory swing is closely given by

$$T_o = 4\sqrt{\frac{R}{g}}\int_0^{\pi/2}\frac{d\theta}{\sqrt{1-k^2\sin^2(\theta)}} \quad with \quad k=\sin\left(\frac{\theta_{pk}}{2}\right)$$
$$T_o = 2\pi\sqrt{\frac{R}{g}}\left[1+\left(\frac{1}{2}\right)^2 k^2+\left(\frac{1\bullet 3}{2\bullet 4}\right)^2 k^4+\left(\frac{1\bullet 3\bullet 5}{2\bullet 4\bullet 6}\right)^2 k^6+\ldots\right]\cong 2\pi\sqrt{\frac{R}{g}} \quad (1)$$

in which R is the pendulum's length as shown in Figure 1, g is the acceleration of gravity (taken to be 32.174 ft/s^2 or equivalently 9.81 m/s^2), and θ_{pk} is the maximum angular excursion of the swing. My grandfather clock exhibits a pendulum period of two seconds thereby implying a pendulum length of about 39.12" from (1). This is convenient given the one pulse-per-second rate directly available from my GPS receiver.

The top-level plan for this project is sketched out in Figure 2. The plan has admittedly grown somewhat as I began putting pen to paper, in large part because I already had most of the components in hand. The main ingredients of the plan include the following:

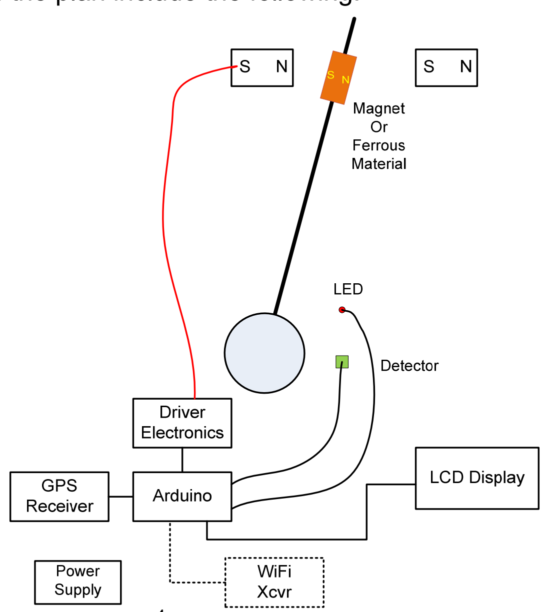

Figure 2 Top-level sketch of planned effort[4]

- use the 1 PPS output from the GPS receiver as the master time source
- read out GPS time and display on the LCD screen when requested by the user
- use an LED source and detector arrangement to precisely measure the amplitude and duration of each individual pendulum swing
    - derive injection locking status
    - derive swing statistics
- adaptively modify the amount of electromagnetic force impressed on the pendulum's arm
    - primarily modified by changing the number of electromagnetic pulses applied over every time length T and or the pulse-length applied.
    - time duration and magnetic force applied by each pulse should be small enough so as to only perturb the pendulum's motion very slightly rather than abruptly jar it in any way.

[4] From U27449 Pendulum Clock Figures.vsd.

- the free-running accuracy of the stand-alone pendulum clock is presently on the order of one or two seconds per day, so the electronics can be put into sleep mode most of the time if desired.
- I may add a WiFi module in later to make it easy to back-haul telemetry information for additional post-analysis, but I predict I will need to exhaust my present list of project before getting further obsessed with this effort.

1.1 Mathematical Basis

The inclusion of electromagnetic forces upon the pendulum's arm motion in Figure 1 changes the mathematics originally discussed in [1]. This figure must be modified as shown in Figure 3. The horizontal force imposed on the pendulum arm is a function of the instantaneous angle φ because each pulse has a finite time-duration during each pendulum period, and the magnetic lines of flux have curvature making the net applied electromagnetic force also a function of φ. The pendulum's arm is assumed to have no mass as in [1]. Since mass does not enter into formula for the period (1), this would seem to be a plausible assumption to make.

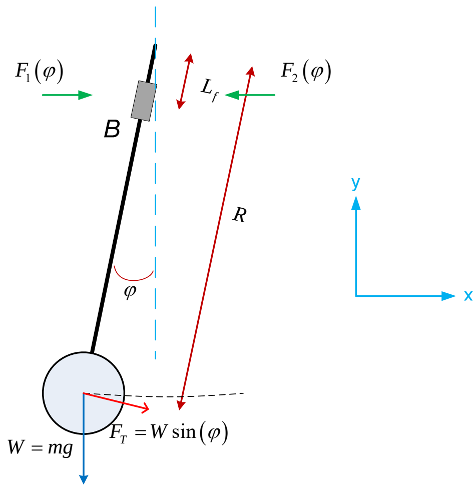

Figure 3 Modified pendulum diagram[5]. The small magnet (or ferrous metal) piece is denoted by *B*. Angle φ is taken as positive when it extends to the left-side of vertical.

It is convenient to think of the pendulum bob's motion in terms of motion along the fixed radius *R* where its angle relative to vertical is given as a function of time by φ. The tangential force perpendicular to *R* created by the weight of the bob is given by

$$F_T = -mg\sin(\varphi) \tag{2}$$

and the associated force-moment is given by $M(\varphi) = F_T(\varphi)R$. This distinction is helpful because the electromagnetic forces F_1 and F_2 are applied[6] at length L_f thereby having different force-moments. The analysis is simplified if all forces acting upon the bob can be expressed as equivalent forces acting on the center of the bob. To this end, and to accommodate using ferrous material or a small magnet at length L_f, the new forces F_1 and F_2 will only be used to *attract* (rather than repel) and their respective force-

[5] From U27449 Pendulum Clock Figures.vsd.
[6] F_1 refers to an attractive force asserted from left of the pendulum, F_2 for the right side.

moments are denoted here by $L_f F_n(\varphi)$. If *B* is ferrous material, only F_1 or F_2 will be present at any instant in time.

Summing the forces at work on the pendulum bob leads to

$$F_{All} = F_T(\varphi) - \left(\frac{L_f}{R}\right)\left[F_1(\varphi) + F_2(\varphi)\right]\cos(\varphi) \tag{3}$$

From Newton's laws of motion, this tangential force must be associated with a tangential acceleration which can be written as

$$\begin{aligned} F_{All} &= ma_T = m\left(\frac{dv_T}{dt}\right) = m\frac{d}{dt}\left(R\frac{d\varphi}{dt}\right) \\ &= mR\frac{d^2\varphi}{dt^2} \end{aligned} \tag{4}$$

Combining (2), (3), and (4) produces the differential equation of motion as

$$\frac{d^2\varphi}{dt^2} + \left(\frac{g}{R}\right)\sin(\varphi) + \left(\frac{L_f}{mR^2}\right)\left[F_1(\varphi) + F_2(\varphi)\right]\cos(\varphi) = 0 \tag{5}$$

Pendulum clocks are generally designed to operate with $|\varphi|$ limited to less than roughly 5° in which case the cosine term is always greater than 0.996. Similarly, the small-angle approximation can be invoked for $\sin(\varphi)$ and (5) can be simplified to

$$\frac{d^2\varphi}{dt^2} + \left(\frac{g}{R}\right)\varphi + \left(\frac{L_f}{mR^2}\right)\left[F_1(\varphi) + F_2(\varphi)\right] = 0 \tag{6}$$

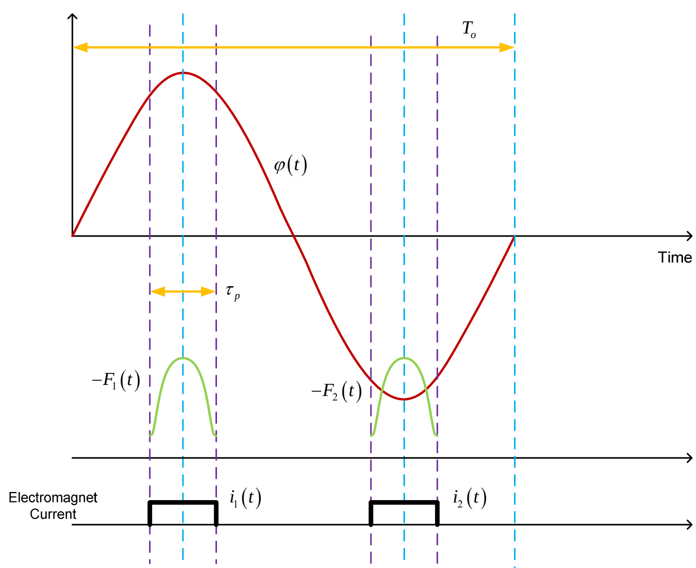

Figure 4 Timing diagram associated with (6)

A timing diagram associated with (6) is shown in Figure 4. For all practical purposes, the time function $\varphi(t)$ can be taken to be sinusoidal as shown. The diagram shows the relationships between the pendulum's swing and applied electromagnetic pulses/forces when the pendulum's swing is in perfect phase lock with the electronics. The current pulses applied to each electromagnet are rectangular with a duty factor of $d = \tau_p / T_o$ whereas the forces exerted on *B* are nonlinear as shown.

Even though the electromagnetic-applied forces are nonlinear pulses as shown in Figure 4, the mechanical coupling is purposely very small and the pendulum's motion acts like a very narrow time filter. Consequently, only the coupling action's Fourier component at the fundamental frequency T_o^{-1} is of any consequence. Taking this train of thought one step further, it suffices to approximate

$$F_1(\varphi) + F_2(\varphi) \approx \gamma\varphi \tag{7}$$

in (6) where γ is the amplitude of the Fourier term. Substituting this into (6) leads to

$$\frac{d^2\varphi}{dt^2} + \left(\frac{g}{R}\right)\varphi + \left(\frac{L_f}{mR^2}\right)\gamma\varphi = 0 \tag{8}$$

and upon collecting terms,

$$\frac{d^2\varphi}{dt^2} + \left(\frac{g}{R}\right)\left(1 + \frac{L_f\gamma}{mgR}\right)\varphi = 0 \tag{9}$$

The pendulum's frequency is consequently modified from (1) to

$$f_o' = f_o\sqrt{1 + \frac{L_f\gamma}{mgR}} \cong f_o\left(1 + \frac{L_f\gamma}{2mgR}\right) \tag{10}$$

This result provides the basis for modifying the pendulum's frequency and phase. This next section considers the questions surrounding injection locking of the pendulum to the applied corrective forces.

1.2 Adler's Equation

As already mentioned in §1, Huygens can be credited with first identifying injection locking between mechanical oscillators. Adler was the first to capture this locking mechanism mathematically in terms of an electronic oscillator. While Adler's theory will in no direct way provide information about the needed electromagnetic pulse length or 1-out-of-*N* application details, the mathematics do provide insight into the underlying factors which matter the most. In the interest of brevity, however, I will leave this topic to the interested reader.

1.3 Pendulum Motion in More Detail

Two rather important points need to be made about (5). First of all, there has been no inclusion of the very small energy *bump* provided each cycle by the clock's escapement mechanism. Without this small addition of energy, the pendulum's swing would eventually decay to nothing. The swing amplitude decay can be attributed to mechanical imperfections in the flat steel spring which connects the pendulum arm to the clock mechanism, and other losses such as air resistance. If the pendulum is not moving (either at the bottom of its swing or one of the two peaks), it is reasonable to assume there are no losses being incurred and it makes sense to approximate the (missing) loss term as $\alpha \frac{d\varphi}{dt}$. The homogenous differential equation is then given by

$$\frac{d^2\varphi}{dt^2} + \alpha \frac{d\varphi}{dt} + \left(\frac{g}{R}\right)\sin(\varphi) = 0 \tag{11}$$

The pendulum's swing can be numerically computed as discussed in §5.

1.4 How Much Force to Apply?

The original idea for coupling the synchronizing force into the pendulum was as sketched in Figure 3. For my grandfather clock, however, the most noninvasive means to inject this energy is horizontally near the peak of the pendulum bob's swing because

- the bob is ferrous in nature and no change need be made to the original pendulum; the synchronizing electromagnetic force(s) can be impressed upon it directly.
- power-wiring can be done at the base of the clock rather than close to the clock's gear mechanism where maneuvering around the clock chimes would have also been inconvenient
- measuring the pendulum's swing-amplitude can be done with greater precision

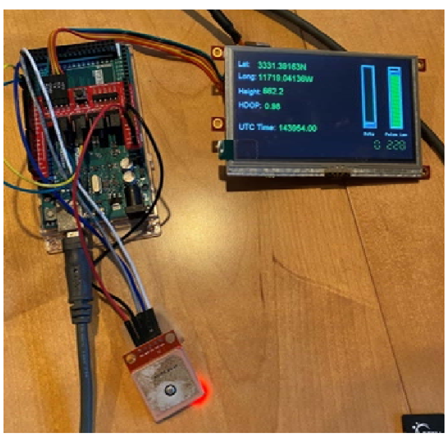

Figure 5 Arduino Mega2560, LCD, and GPS receiver during software development. Latitude, longitude, Height, HDOP, and UTC are also read out continually

Figure 6 My grandfather clock

Figure 7 View inside the pendulum swing area

Admittedly, the most accurate swing interval time measurements would normally be made at the location where the pendulum is swinging the fastest (bottom of the arc-swing), but this would also detract from the clock's normal appearance.

Each electromagnetic pulse-force applied must have some finite pulse-duration associated with it. Having longer pulses reduces the peak-current required and inductive kick-back from quickly turning off the current each cycle. Positioning the electromagnet(s) and amplitude sensor(s) at the arc-swing limits co-locates all of the apparatus, and the control law for synchronization can be designed to drive the pendulum's phase such that its swing in time is symmetrically centered on each electromagnetic pulse. In principle, one LED emitter/detector pair can then be used to measure both swing amplitude and swing phase and period.

1.4.1 Electromagnets

Electromagnet design details were not as straight forward as I had hoped. Off-the-shelf electromagnets use high-μ core materials which dramatically increase the local magnetic field strength but have a correspondingly weak field even a few tenth of an inch away. The as-advertised holding-power is only realized when the electromagnet is literally placed in contact with the ferrous material being attracted. This constraint is unacceptable for this application because any collision between the electromagnet and pendulum bob would seriously impact the time-keeping precision. Another approach was sought.

To this end, I explored coreless wire loops as discussed in §4. Although the required attractive force between the electromagnet and pendulum bob is very small, the fields created even with several thousand turns of magnet wire were small enough to raise my concern.

I subsequently looked into methods by which the fairly weak magnetic field(s) could be concentrated or focused. This effort led me to US Patent 5,929,732 (July 27, 1999) titled “Apparatus and Method for Amplifying a Magnetic Beam.” The main concept is illustrated in Figure 8 from the patent.

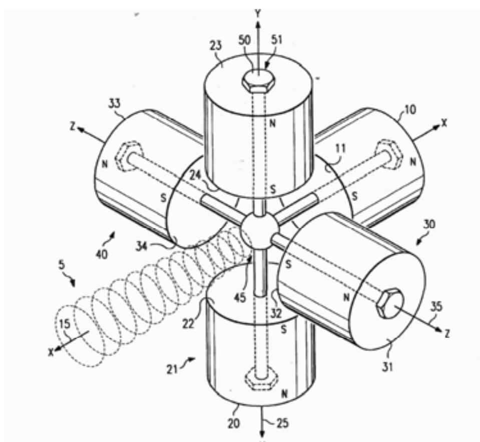

Figure 8 Focusing a magnetic beam from US Patent 5,929,732. All of the metal supporting bolts are non-ferrous in nature.

The implementation I have chosen for my pendulum project is to use high-intensity neodybium magnets for the four coplanar elements while only making the x-axis element in Figure 8 an electromagnet.

2 Part II

Part II will address the remaining elements of this project including:

- fabrication of the focused electromagnet and some measures of its performance
- LED emitter/detector for measuring the pendulum's period and swing amplitude
- additional details on the electronics portion shown in Figure 5
- adopted pendulum control law
- final packaging and performance of the grandfather clock

3 References

1. J.A. Crawford, "Pendulums and Elliptic Integrals," 2003.
2. ___________, *Advanced Phase-Lock Techniques*, Artech House, 2008.
3. Robert Adler, "A Study of Locking Phenomena in Oscillators," *Proc. IRE*, June 1946, U10751.
4. David Halliday and Robert Resnick, *Fundamentals of Physics*, John Wiley & Sons, 1970.

4 Appendix: Electromagnetics

I initially assumed I would purchase a simple low-cost electromagnet for the project, but this approach is not viable because these electromagnets always use a high-μ core material which tightly concentrates the magnetic flux lines only near the surface of the electromagnet. If a ferrous object is placed in contact with the electromagnet, the holding/gripping strength can be very high but the field strength drops off dramatically even a fraction of an inch away. For my pendulum application, I do not want the pendulum's swing to come this close to the electromagnet because a slight error in the swing amplitude could easily cause the pendulum to collide with the electromagnet thereby dramatically impairing the time precision. Rather, the electromagnet needs to have an appreciable *reach* of 1 to 2 inches.

I subsequently considered producing the needed magnetic field by using a straight forward multi-turn wire loop. A circular single-turn loop is shown in Figure 9 where it is assumed the pendulums swing would be in the *x*-axis direction. Reference [4] gives the $B_{\|}$ field magnitude as

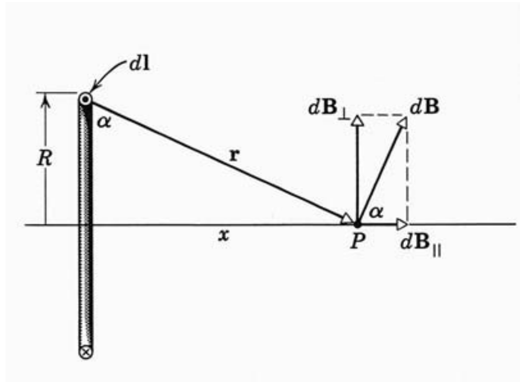

Figure 9 Side-view of circular wire loop[7] and its associated axial magnetic field ***B***

$$B_{\parallel} = \frac{\mu_o i R^2}{2\left(R^2 + x^2\right)^{3/2}} \tag{12}$$

for measurement positions ***P*** along the x-axis where *i* is the single-loop current in Amperes. Since the pendulum-bob's diameter is about 6" and it is desirable from a field-strength perspective for the edge of the bob to be able to swing up to the center point of the wire loop, ***R*** = 3" will be assumed. The normalized version of (12) is plotted in Figure 10 and still shows good field strength at 1" from the center of the circular coil. So long as the actual value of the field strength is sufficient to suitably attract the ferrous pendulum bob, this design approach should be far superior to a purchased high-μ electromagnet.

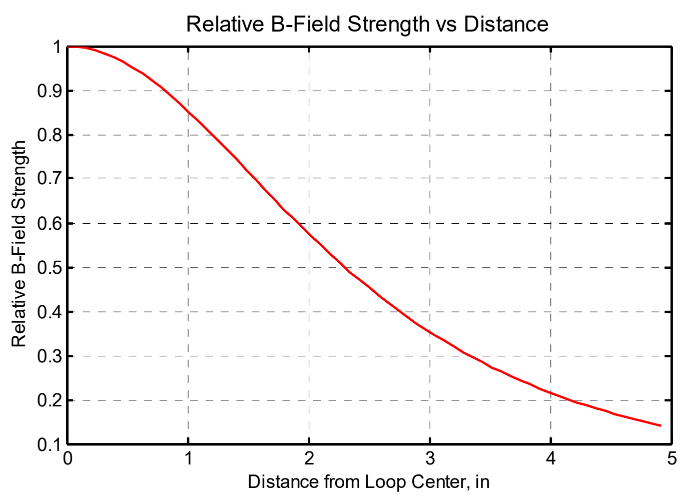

Figure 10 Relative field strength (12) as point *P* in Figure 9 is moved along the x-axis. The field strength is still 85% of its maximum possible value at x = 1".

4.1 Loop Shape

The previous discussion suggests the *reach* of a classical circular multi-turn loop should be rather good. To further the discussion, one question worthwhile addressing is whether a circular loop (rather than an elliptical) loop is even better. To this end, the Biot-Savart Law can be used to investigate this question.

The Biot-Savart Law may be written in differential vector form as

[7] Figure 30-16 of [4].

$$d\vec{B} = \frac{\mu_o i}{4\pi}\frac{d\vec{l}\times\vec{r}}{r^3} \tag{13}$$

where vector quantites $\vec{r}$ and $\vec{r}$ are as denoted in Figure 11. For points *P* inside the wire loop, there are simplified forms of (13) which can be applied, but I chose to retain the cross-function as is.

The wire loop shape I considered were all ellipses with different eccentricity values but with the same vertical opening to permit the pendulum bob to swing slightly into the loop if desired or needed. The actual dimensions considered are shown in Figure 12.

Next, the Biot-Savart Law was computed for the perimeter of each ellipse at a point at the center of the ellipses, but displaced out of the plane by different values of *d*, and the results compared as shown in Figure 13. In this figure *d* = 0 corresponds to the point at the center of the ellipse and in the same plane as the wire loops. The *d* = 0 curve results in a noticeably stronger *B*-field, thereby substantiating having the pendulum bob swing at least a small amount into the plane of the wire loops.

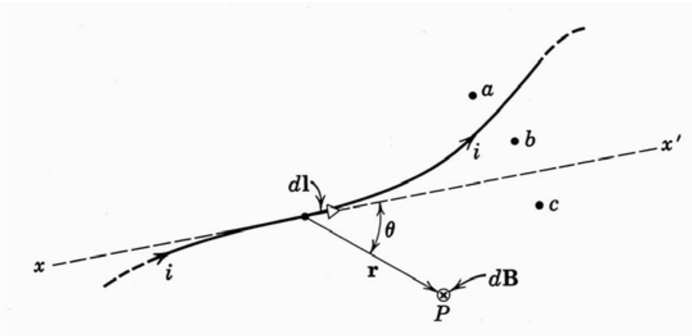

Figure 11 Application of differential Biot-Savart Law[8]

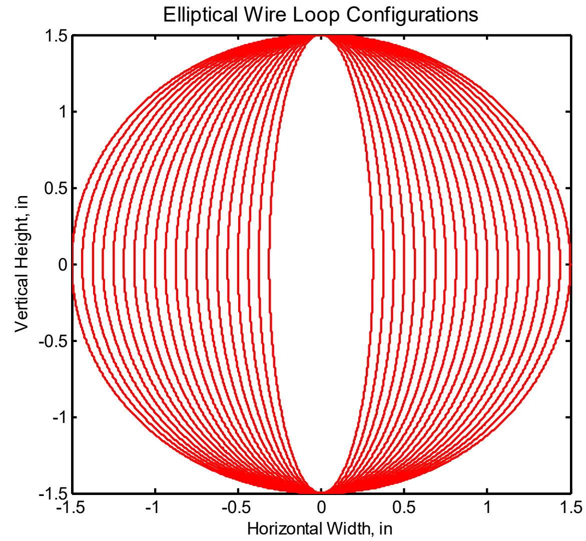

Figure 12 Elliptical wire loop configurations[9]

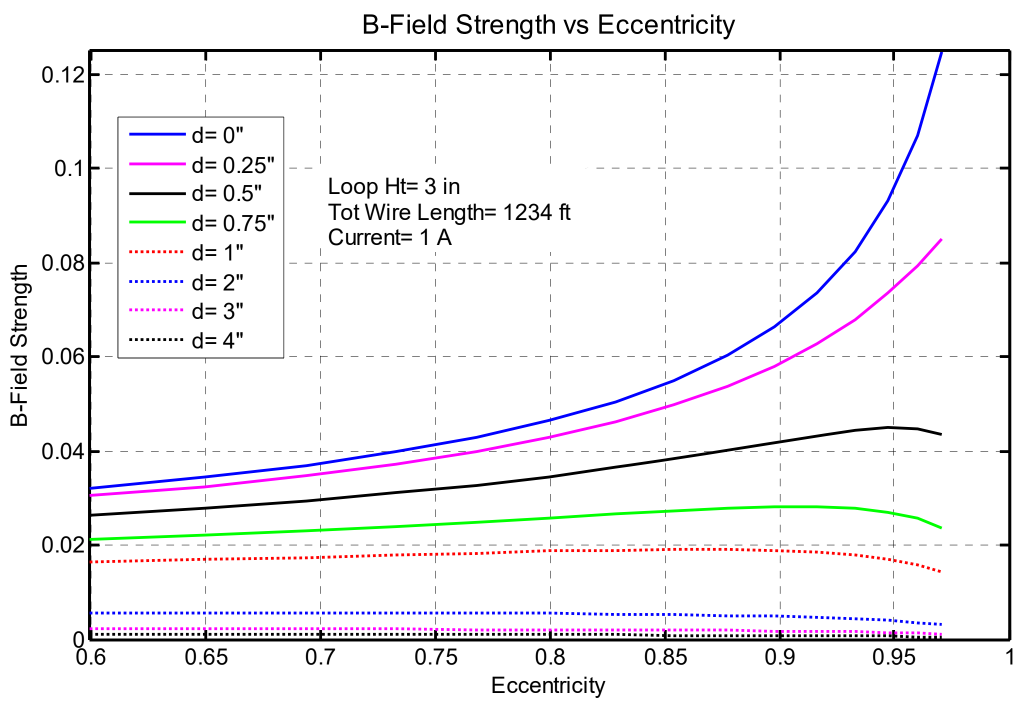

Figure 13 Normalized magnetic field strength[10] for the loop shapes shown in Figure 12 where the spatial point for the field evaluation was (x,y,z)= (0, 0, *d*) with *d* in inches

8 Figure from Figure 30-15 of [4]
9 u27501_pendulum_electromagnet.m.
10 u27504_pendulum_electromagnet.m.

5 Appendix: Pendulum Swing Simulation

The nonlinear differential equation for pendulum motion (11) can be handled with good accuracy using implicit numerical integration (specifically, *Backward Euler*). To this end, let

$$u = \frac{d\varphi}{dt} \qquad \frac{du}{dt} = -\alpha u - \frac{g}{R}\sin(\varphi) \tag{14}$$

Then define the equations in terms of first-order difference equations as

$$\frac{d\varphi}{dt} \Rightarrow \frac{\varphi_n - \varphi_{n-1}}{dt} \qquad \frac{du}{dt} \Rightarrow \frac{u_n - u_{n-1}}{dt} \tag{15}$$

Solving (15) for φ_n and u_n leads to

$$\varphi_n = dt\,u_n + \varphi_{n-1} \qquad u_n = \frac{u_{n-1} - \frac{dt\,g}{R}\sin(\varphi_n)}{1 + \alpha\,dt} \tag{16}$$

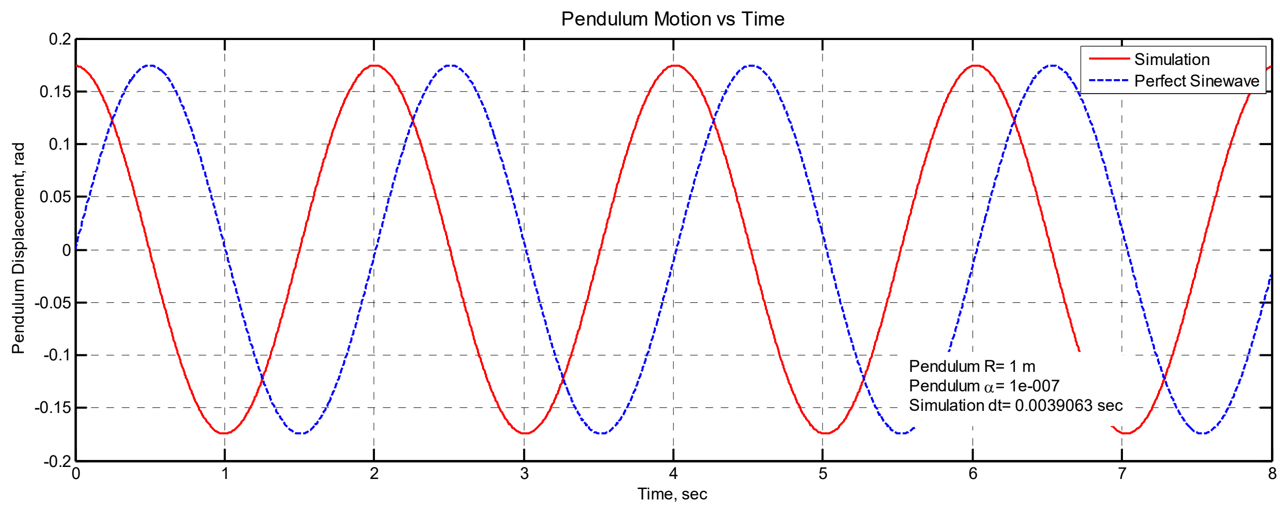

Figure 14 Pendulum swing with negligible loss per period[11]

[11] u27505_pendulum_swing.m.

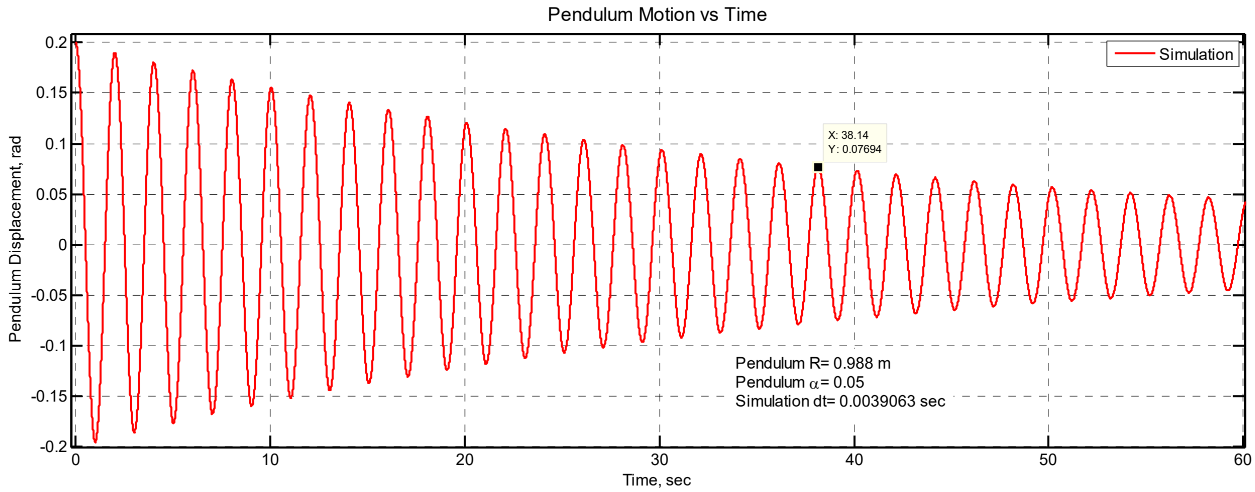

Figure 15 Pendulum swing with much greater loss per period

6 Bill of Materials

- 4D Systems μLCD-43-PT-AR
- Arduino Mega2560
- Ublox NEO-6 GPS Receiver
- Electromagnet
- Electromagnet driver and LED/detector circuitry
- Wall-wart power supply
- Housing
- Electromagnet/circuitry support arm