Bartlett's Bisection Theorem

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Bartlett's bisection theorem is a theorem in electrical engineering that connects symmetrical filter networks with their lattice equivalent form. In the context of this short memo, this theorem will be used to quickly transpose a symmetrical electrical filter operating between equal source and load resistances to a network operating between difference source and load resistances.

Take for instance a normalized 3rd-order Butterworth lowpass filter. The normalized filter values are shown schematically in Figure 1. The S_{21} and S_{11} characteristics for this network are plotted in Figure 2.

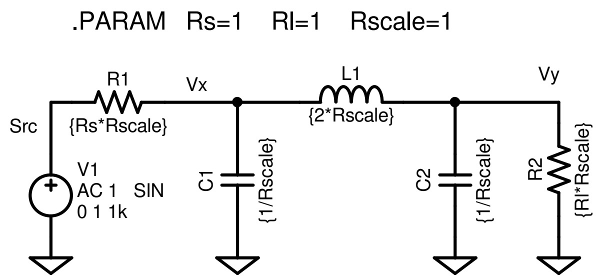

Figure 1 Symmetrical N = 3 Butterworth lowpass filter with normalized values [C1, L1, C2] = [1, 2, 1] and R1 = R2 = 1Ω

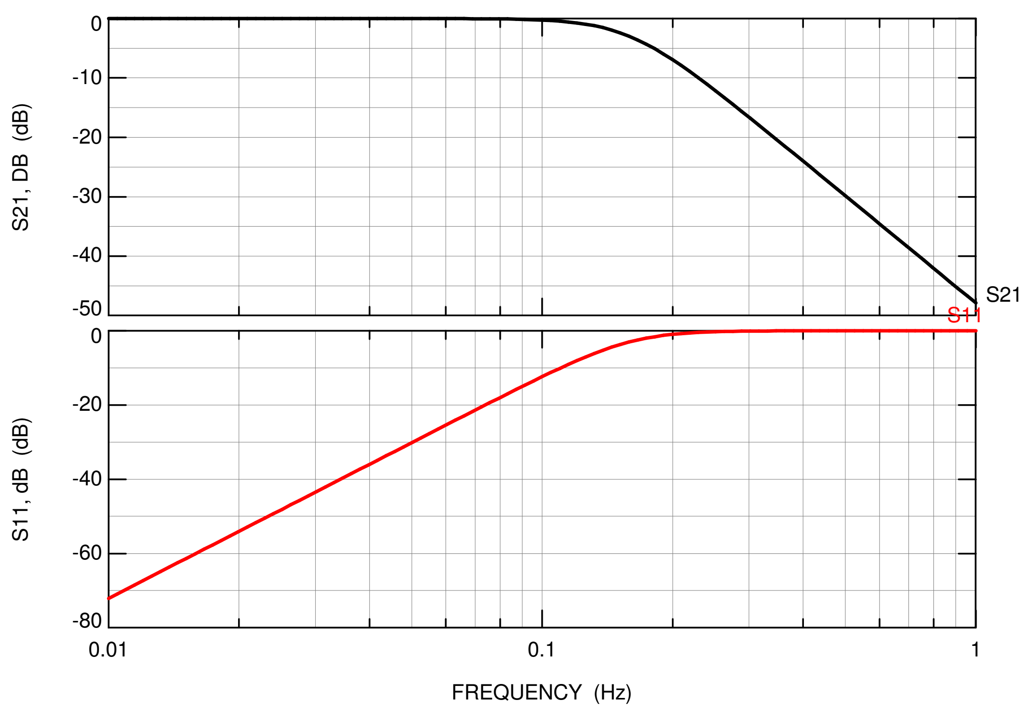

Figure 2 S_{21} and S_{11} for the basic Butterworth lowpass filter shown in Figure 1

Assume now that we would like to operate this filter between a source impedance of 1Ω and a load impedance of 2Ω. Bartlett's theorem directs us to first identify a plane of symmetry as shown in Figure 3. The second step merely involves impedance-scaling all of the circuit elements to the right of the plane of symmetry by the output to input impedance ratio g (= 2 in this example) as shown in Figure 4. In the final step, series and parallel circuit elements are recombined where possible as shown in Figure 5. The Spice analysis schematic for this example is shown in Figure 6. The resultant S_{21} and S_{11} performance characteristics are shown in Figure 7. In this context, S_{21} is defined as 10 $\log_{10}(P_{Delivered} / P_{Available})$.

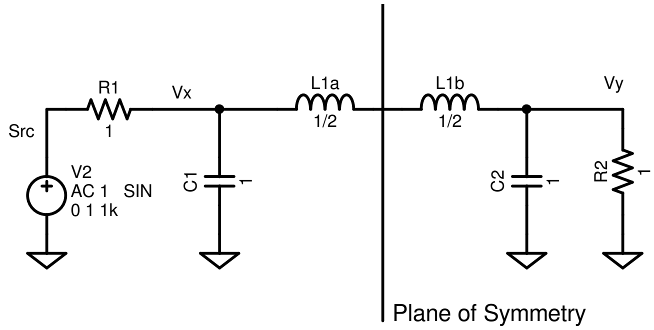

Figure 3 First step in implementing the Barlett Bisection Theorem: identify the plane of symmetry left to right

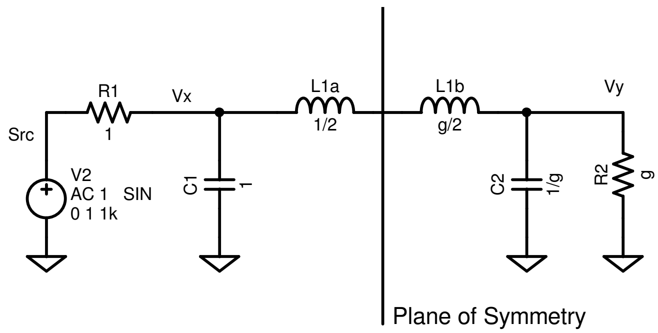

Figure 4 Second step in Bartlett's Bisection Theorem is to impedance-scale all of the components by ratio of the desired output impedance to input impedance ratio (equal to g here)

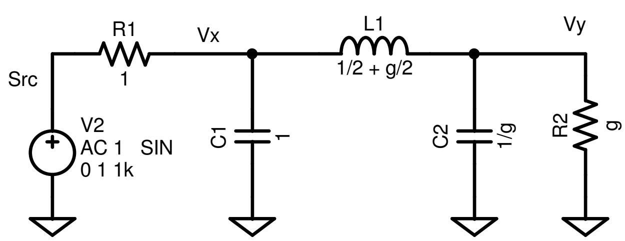

Figure 5 Third and final step in Bartlett's Bisection Theorem is to combine series and parallel circuit elements where possible

.PARAM g=2 L1={Rscale+Rscale*g} C2={1/Rscale/g} R2={Rl*Rscale*g}

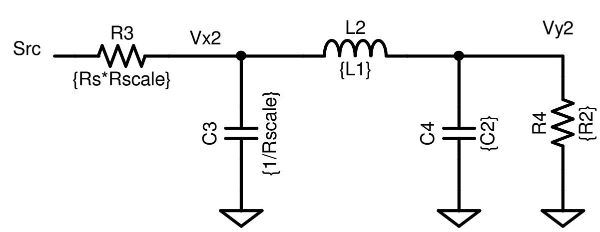

Figure 6 Bartlett theorem transformed Butterworth lowpass filter working between the original source resistance and new load resistance RL = 2

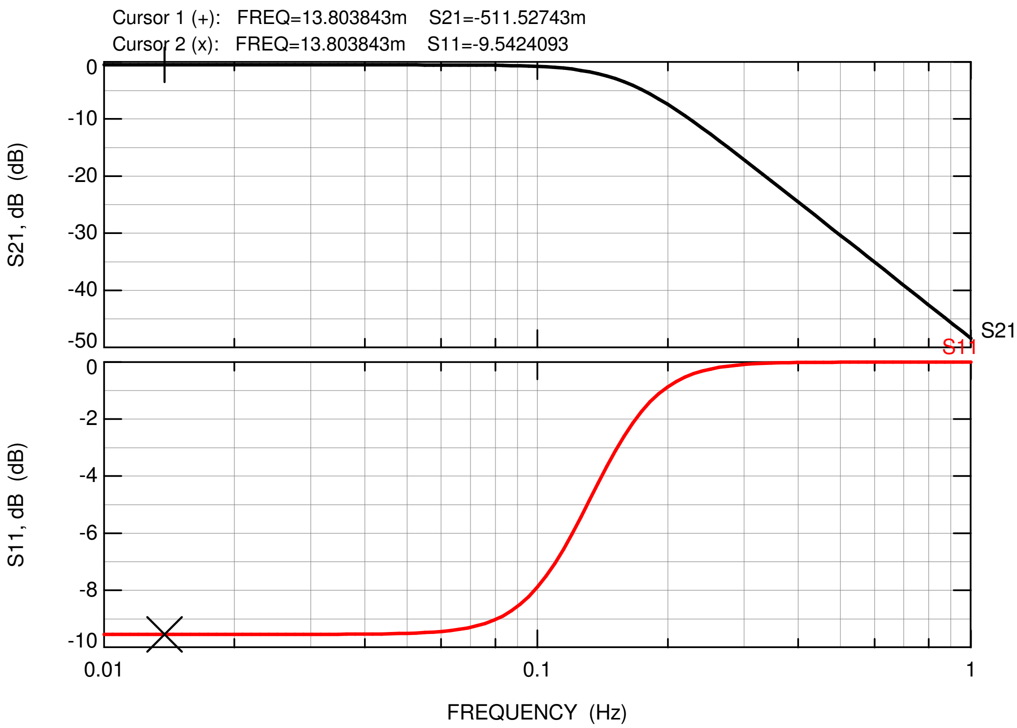

Figure 7 S_{21} and S_{11} for the Bartlett transformed Butterworth lowpass filter shown in Figure 6

Notice that in Figure 7, the desired S_{21} characteristic shape has been perfectly retained as predicted by the theorem, but $|S_{21}| < 0$ dB. The input return loss is, however, quite poor and in fact given by

$$RL_{dB} = 20\log_{10}\left(\left|\frac{g+1}{g-1}\right|\right) \tag{1}$$

and the impedance mismatch is a direct consequence of using the theorem. Similarly, the transducer gain ($|S_{21}|$) at DC is given by

$$G_{DC} = 10\log_{10}\left[\frac{4g}{(1+g)^2}\right] \quad dB \tag{2}$$

Conclusion: The Bartlett Bisection Theorem is a convenient means to convert a passive symmetrical filter to accommodate dissimilar source and load impedance levels but the impedance mismatch represented by (1) and the power loss represented by (2) must also be tolerated if this method is employed. The only alternative is to employ additional passive circuit elements to physically transform the impedance levels as required.

References:

1] Williams, A. and Taylor, F., *Electronic Filter Design Handbook*, 4th ed., McGraw-Hill Book, 2006.
2] www.wikipedia.com, entry for "Bartlett's Bisection Theorem."