Current Flow in a Conductive Metal Sheet

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The ground-return signal is one of the most important signals in any circuit design. Poor grounding will render even the most robust circuit designs, analog and digital, problematic at best.

This short memorandum looks considers the direct-current flow that occurs in a square metal conductive sheet when different portions of its perimeter are tied to an ideal ground. Simplistic brute-force computation methods are used to numerically compute the resulting behavior.

Ohm's Law is mathematically given by[1]

$$j = \sigma \mathcal{E} = -\sigma \nabla \phi \qquad (1)$$

where

j current density (A-cm^{-2})
σ conductivity (mho-cm^{-1})
$\mathcal{E}$ electric field (V/cm)
ϕ electric potential (V).

Taking the divergence of both sides of (1) produces

$$\begin{aligned} \nabla \bullet j = \sigma \nabla \bullet \mathcal{E} &= -\sigma \nabla \bullet \nabla \phi \\ &= -\sigma \nabla^2 \phi \end{aligned} \qquad (2)$$

In the absence of any current sources or sinks, (2) simplifies to the well-known Laplace equation that is given by

$$\nabla^2 \phi = 0 \qquad (3)$$

The solution method employed here only relies on Ohm's Law and a simple application of Kirchhoff's current law. A more

extensive treatment of this solution can be found in Sections 3.09–3.11 of [3].

The square conductive metal sheet that is to be modeled is shown in Figure 1. A voltage source is connected in one corner region with the two opposite corners tied to ground.
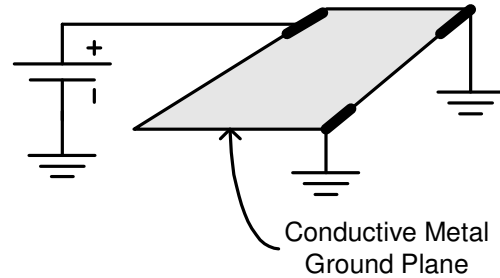
Figure 1 Conductive sheet shown with voltage source at one corner and grounded regions at two other corners.

The conductive sheet is modeled as a grid of discrete resistors as shown in Figure 2. In the case of a uniformly conducting sheet, all of the resistors in the grid have the same value. Otherwise, different values can be used to model non-uniform sheet resistivity.
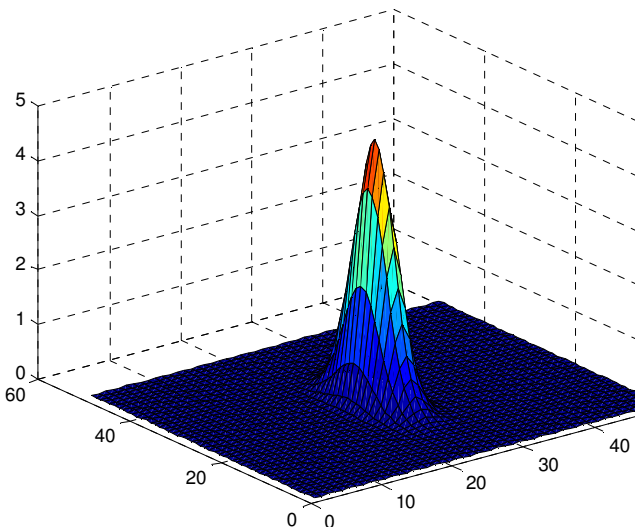
Each internal voltage node is labeled as (n,m) in the figure, and the individual resistors in the grid are denoted by R$_k$.
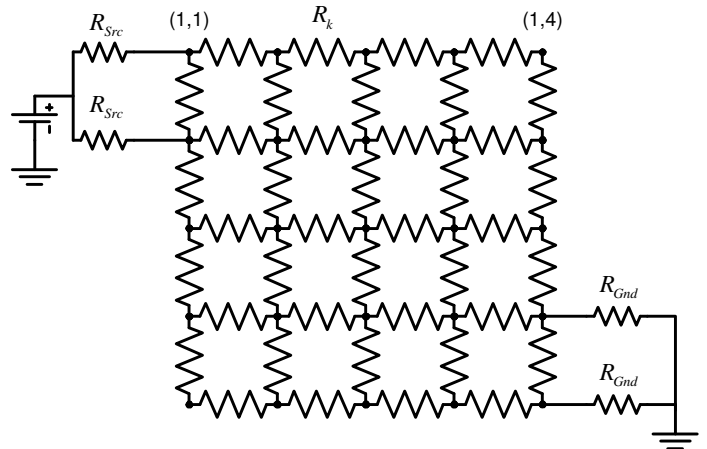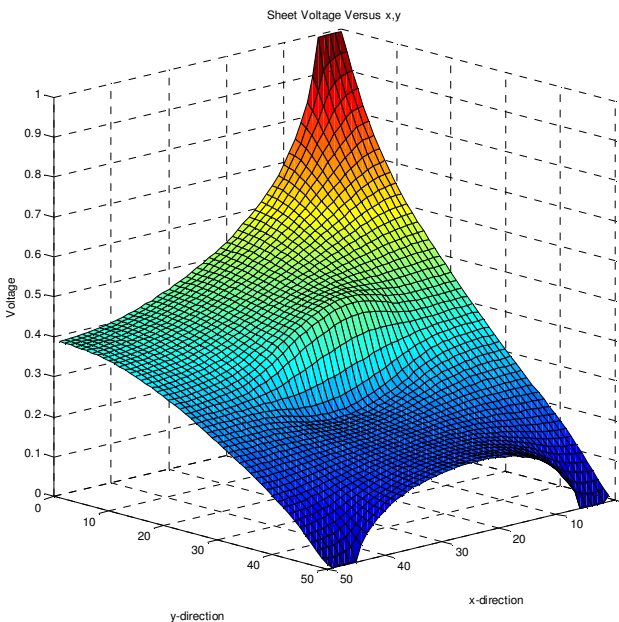

Figure 2 Resistance grid used to model the conductive sheet shown in Figure 1. Two nodes connected to the voltage source, and only one corner region connected to ground.

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[1] Equ. (6.57) of [2].

Two example calculations are shown in the figures below. A resistive grid of 50×50 nodes was used for the calculations with the non-uniform sheet resistivity characteristic shown in Figure 3. When the two opposite corners of the sheet are tied to ground, the resulting voltage profile across the conductive sheet is like that shown in Figure 4.

When a third portion of the sheet's perimeter is tied to ground, the voltage distribution across the sheet changes markedly as shown in Figure 5.

The Matlab code used to create these figures is provided in the appendix in order to make it easy to try other variants of this simple arrangement. Modification of the resistance grid values can be used to illustrate current crowding, breaks purposely put in the ground plane, and other hypothetical cases thereby providing insight into grounding effects.

Figure 3 Non-uniform sheet resistivity characteristic used in the sample computations.
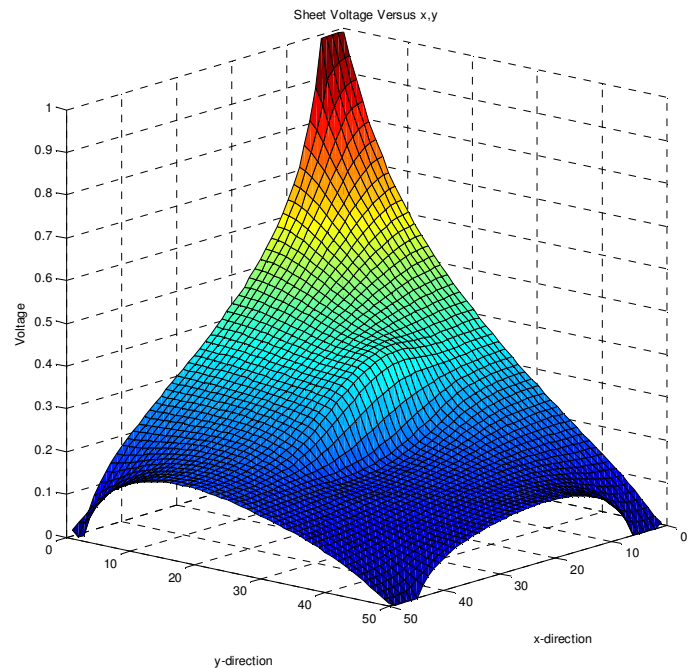
Figure 5 Sheet voltage profile with three corners grounded.

Figure 4 Sheet voltage profile with two corners grounded.

1 References

1. Crawford, James A., *Frequency Synthesizer Design Handbook*, Artech House, 1994.
2. Condon, E.U., and H. Odishaw, *Handbook of Physics*, 2nd ed., McGraw-Hill Book, 1967.
3. Ramo, S., J.R. Whinnery, and T. Van Duzer, *Fields and Waves in Communication Electronics*, John Wiley and Sons, 1965.

Appendix

```
%================= u14066_sheet_bvp.m =============================
%
%   Sheet Resistivity Boundary Value Problem
%
%   J.A. Crawford  Copyright 2008
%
%   Consider a square two-dimensional resistive sheet. Apply voltage
%   at a portion of one edge with short ground segments long other edge(s).
%   Compute the voltage across the sheet.
%   From the computed surface voltage, compute the magnitude of current
%   flow at each (x,y) location in the sheet.
%
%   The sheet is broken up into a mesh of equally-spaced points. Discrete
%   resistors are assumed to be connected between the mesh points. Given
%   all of the resistance values identified, an admittance matrix is
%   assembled from which all of the node-voltages are computed.
%
Ncol= 50;       % Number of points in x-direction in sheet
Nrow= Ncol;     % Number of points in y-direction in sheet
%
dx= 1/Ncol;
%
%   Convert rol,col coordinates to a linear vector index
%
[xx,yy]= meshgrid(1:Ncol, 1:Nrow);
zz= (yy-1)*Ncol+xx;
%
Ix= zeros(Ncol*Nrow,1); % Source current vector for all mesh nodes
%
%   Initialize admittance matrix
%
ymat= zeros(Ncol*Nrow,Ncol*Nrow);
%
%   Position 1 V sources along a portion of one edge
%   Each source has a source resistance of Rsource ohms
%
Nsources= floor( (6/60)*Ncol );
Rsource= 0.001*Nsources*dx;
for ii=1:Nsources
    kk= zz(1,ii);
    Ix( kk )= 1/Rsource;
    ymat(kk,kk)= 1/Rsource;
end
%
%   Tie part of one edge to ground
%
N0= Nrow*Ncol;
Nties= floor( (6/60)*Ncol );    % Number of ties to ground
for ii=1:Nties
    kk= zz(Nrow,Ncol)-ii;
    ymat(kk,kk)= 1/(0.00001*Nties*dx);
end
for ii=1:Nties
    kk= zz(Nrow,1) + ii;
```

```matlab
        ymat(kk,kk)= 1/(0.00001*Nties*dx);
end
%
%   Add one more ground-tie region along an edge if desired
%
if( 1 )
    for ii=1:2
        kk= zz(1,Ncol)-ii;
        ymat(kk, kk)= 1/(0.00001*Nties*dx);
    end
end
%
%   Pick choice of different sheet resistivity functions
%
%   rsw= 1    constant resistivity
%        2    large resistance edge in center of sheet
%        3    large conductivity edge in center of sheet
%
rsw= 2;
switch(rsw)
    case 1
        Rx= @(ii,jj) 2.5;
    case 2
        Rx= @(ii,jj) (2.5 + 240.099*exp(...
                    -(0.2*(ii-1*Nrow/2).^2 + 0.03*(jj-2*Ncol/4).^2) ) )*dx;
    case 3
        Rx= @(ii,jj) (25 - 24.999*exp(...
                    -(0.02*(ii-1*Nrow/2).^2 + 0.003*(jj-2*Ncol/4).^2) ) )*dx;
end
%
%   Compute list of discrete resistor values and node connections
%
rlist_length= (Ncol-1)*Nrow + (Nrow-1)*Ncol;
rlist= zeros(rlist_length, 3);      % R value, p-node, n-node
%
%   Assemble list of discrete resistors between nodes in mesh
%
%   Horizontally connected R's
%
rr= 1;
for rw=1:Nrow
    for cl=1:Ncol-1
        rlist(rr,1)= Rx(rw,cl);
        rlist(rr,2)= zz(rw,cl);
        rlist(rr,3)= zz(rw,cl)+1;
        %
        rr= rr+1;
    end
end
%
%   Vertically connected R's
%
for cl=1:Ncol
    for rw=1:Nrow-1
        rlist(rr,1)= Rx(rw,cl);
        rlist(rr,2)= zz(rw,cl);
        rlist(rr,3)= zz(rw,cl)+Ncol;
```

```matlab
        rr= rr+1;
    end
end
%
%   Assemble Admittance Matrix
%
for rr=1:rlist_length
    Rval= rlist(rr,1);
    pnode= rlist(rr,2);
    nnode= rlist(rr,3);
    ymat(pnode,pnode)= ymat(pnode,pnode) + 1/Rval;
    ymat(nnode,nnode)= ymat(nnode,nnode) + 1/Rval;
    ymat(pnode,nnode)= ymat(pnode,nnode) - 1/Rval;
    ymat(nnode,pnode)= ymat(nnode,pnode) - 1/Rval;
end
%
%   Solve for Node Voltages
%
vs= ymat\Ix;
vgrid= zeros(Nrow,Ncol);     % Computed mesh node voltages
%
%   Re-assemble node voltage vector into node voltage mesh
%
ii= 1;
for rw= 1:Nrow
    for cl= 1:Ncol
        vgrid(rw,cl)= vs(ii);
        ii= ii+1;
    end
end
%
figure(1);
clf;
surf(vgrid);
title( 'Sheet Voltage Versus x,y' );
xlabel( 'x-direction' );
ylabel( 'y-direction' );
zlabel( 'Voltage' );
%
figure(2);
clf;
contour(vgrid);
grid on;
xlabel( 'x-direction' );
ylabel( 'y-direction' );
title( 'Voltage Contours' );
%
if( rsw ~= 1 )
    Resistance= Rx(xx,yy);
    figure(3);
    surf(Resistance);
    grid on;
end
```