A Cursory Look at Satellite Doppler with Type-3 PLL Tracking

James A. Crawford

1. Introduction

An estimate of the orbital doppler that can occur for a fixed ground station can be obtained by considering the simple planar circular orbit case that is shown in Figure 1. In this case, the earth's radius is given by R and the satellite's altitude above the earth's surface is denoted by H. The instantaneous line-of-sight (LOS) range between the ground station and the satellite is given by L.

In the following discussions, the acceleration due to gravity is taken to be g = 9.81 m/s^2 and the earth's radius is assumed to be R= 6437 m. The orbital velocity can be quickly computed by simply equating the radial acceleration to the acceleration due to gravity which leads to

$$\frac{v_c^2}{R+H} = g \tag{1}$$

For low earth orbits where H << R, asymptotically we have

$$v_c \to \sqrt{gR} \tag{2}$$

which equates to 7.95 km/s or approximately 17,200 miles per hour.

The quantity of interest here as far as doppler is concerned is the time-rate of change of the LOS distance L. In this simple model, the x and y coordinate values are used to denote the satellite's planar position at any given instant in time. Clearly for a circular orbit, we have

$$\begin{aligned} x(t) &= (R+H)\cos(\gamma t) \\ y(t) &= (R+H)\sin(\gamma t) \end{aligned} \tag{3}$$

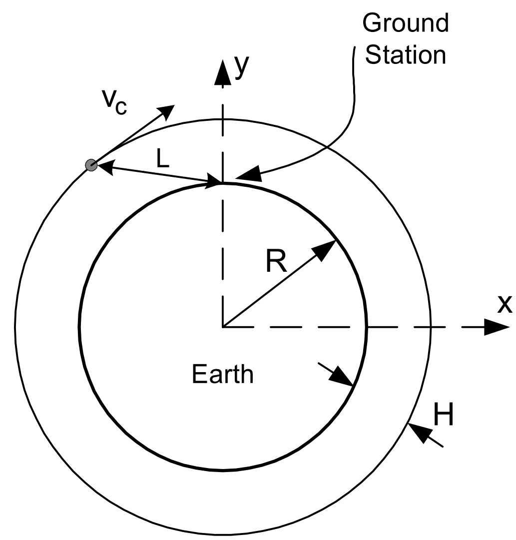
and

$$\gamma = \frac{v_c}{R+H} \tag{4}$$

The distance L is then given by

$$L(t) = \sqrt{x^2(t) + [y(t) - R]^2} \tag{5}$$

Figure 1 Planar Circular Orbit for a Ground Station



The direct LOS velocity as viewed by the ground station is given by the time derivative of (5) as

$$\begin{aligned} v_r(t) &= \frac{dL(t)}{dt} \\ &= \frac{1}{2}\left\{x^2(t) + [y(t)-R]^2\right\}^{-1/2} \\ &\times \left\{2x(t)\frac{dx}{dt} + 2[y(t)-R]\frac{dy}{dt}\right\} \end{aligned} \tag{6}$$

and the time-derivatives are easily substituted in by making use of (3). It turns out that the ratio between the ideal circular orbital velocity and the escape velocity (H << R) is $\sqrt{2}$. Therefore, the worst-case doppler that we can expect to see in practice will at most be approximately 1.4 times that predicted by the circular-orbit theory.

The satellite's (x,y) coordinates are given by the sine and cosine functions for the circular orbit as shown in Figure 2. The preceding equation (5) was used to compute the LOS slant range to the satellite as shown in Figure 3, and equation (6) was used to compute the LOS radial velocity to the satellite. Since doppler frequency is directly related to the satellite's LOS radial velocity, it is a simple matter to take the results in Figure 4 and compute the resulting doppler profile as seen by the ground station.

In Figures 2 through 4, time-axis limits are shown corresponding to the rising and setting of the satellite with respect to the ground station's local horizon. In the example adopted here, these times are given by

$$\begin{aligned} T_{rise} &= \gamma^{-1} \arcsin\left(\frac{R}{R+H}\right) \\ T_{set} &= \gamma^{-1}\left[\pi - \arcsin\left(\frac{R}{R+H}\right)\right] \end{aligned} \qquad (7)$$

These equations can be found by using the geometric limits as given by Figure 1.

Determination of type Type-3 PLL's tracking behavior when subjected to this orbital doppler requires that some additional details be developed. In general, the doppler profile contains some higher-order terms that the Type-3 PLL cannot track-out perfectly thereby leading to some residual tracking bias or error.

Figure 2 Satellite (x,y) Coordinates for H= 700 km

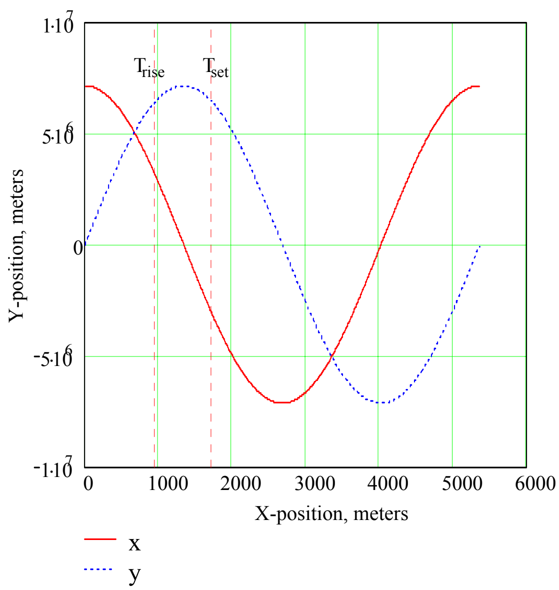


Figure 3 LOS Slant Range to Satellite Versus Time (H= 700 km)

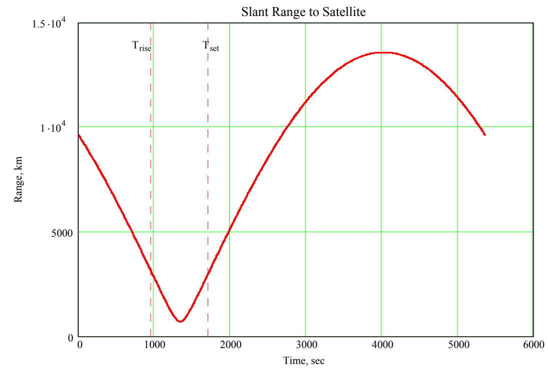
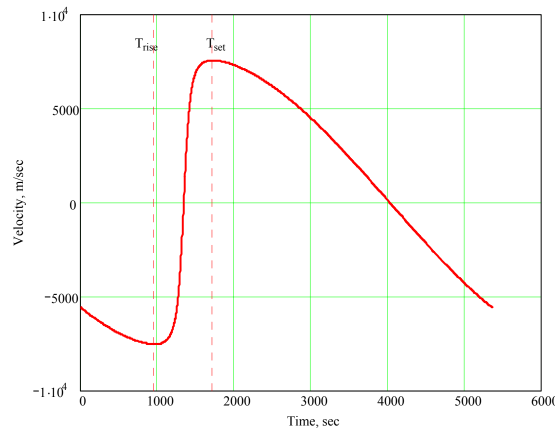


Figure 4 LOS Velocity with Respect to Ground Station (H= 700 km)



2. Overview of Type-3 PLLs

There is not a large amount of literature immediately available on Type-3 PLLs, so some review of the basic tenants is appropriate. The general open-loop gain transfer function for a Type-3 PLL is given by

$$G_{OL}(s) = K\frac{(1+s\tau_2)(1+s\tau_3)}{s(\tau_1 s)^2} \qquad (8)$$

Generally, the time constants τ_2 and τ_3 are chosen to be equal, so this simplifies the open-loop gain function to

$$G_{OL}(s)=\frac{K}{s}\left(\frac{1+s\tau_2}{s\tau_1}\right)^2 \tag{9}$$

The phase of the open-loop gain function is clearly given by

$$\angle G_{OL}(s)=-\frac{3\pi}{2}+2\tan^{-1}(\omega\tau_2) \tag{10}$$

Obviously, $\angle G_{OL}(s) = -\pi$ for $\omega\tau_2 = 1$ which means that this situation occurs for $\omega = \tau_2^{-1}$. The gain at this critical frequency is given by

$$G_M = 2K\tau_2\left(\frac{\tau_2}{\tau_1}\right)^2 \tag{11}$$

A positive gain margin results if and only if $G_M > 1$.

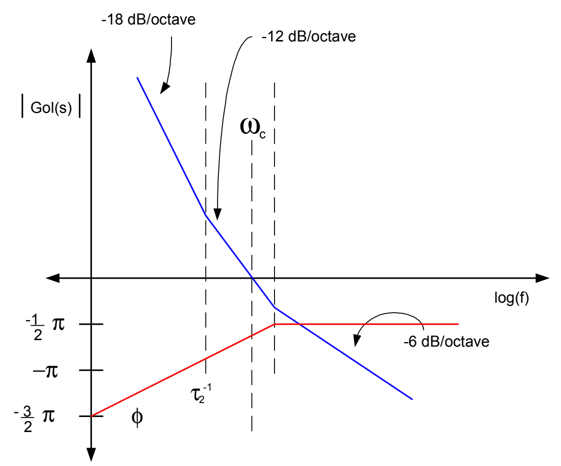
For frequencies $\omega > \tau_2^{-1}$, the control loop is very similar to a classical Type-2 loop. To first-order, for $\omega > \tau_2^{-1}$,

$$\left|G_{OL}(s)\right| \approx \left|\frac{K}{s}\left(\frac{\tau_2}{\tau_1}\right)^2\right| \tag{12}$$

so the unity-gain cross-over frequency (ω_c) is approximately given by

$$\omega_c \approx K\left(\frac{\tau_2}{\tau_1}\right)^2 \tag{13}$$

Figure 5 Open-Loop Gain Characteristic for Type-3 PLL



From earlier, the gain margin was given by (11) and therefore,

$$\omega_c \approx \frac{G_M}{2\tau_2} \tag{14}$$

The exact result for ω_c can be found by solving

$$\left|\frac{K}{j\omega}\left(\frac{1+j\omega\tau_2}{j\omega\tau_1}\right)^2\right| = 1$$

or

$$\omega^3 - K\left(\frac{\tau_2}{\tau_1}\right)^2\omega^2 - \frac{K}{\tau_1^2} \equiv 0 \tag{15}$$

Given an initial estimate for the true solution (denoted by ω_x), a root-polishing recursion can be applied as

$$\omega_x' = \omega_x + d\omega$$

where

$$d\omega = \frac{-\omega_x^3 + K\left(\frac{\tau_2}{\tau_1}\right)^2\omega_x^2 + \frac{K}{\tau_1^2}}{3\omega_x^2 - K\left(\frac{\tau_2}{\tau_1}\right)^2 2\omega_x} \tag{16}$$

Comment: Use of K <u>and</u> τ_1 is redundant. If we simply assign $K_x = K/\tau_1^2$, then the equation to be solved simplifies to

$$\omega^3 - K_x(\tau_2\omega)^2 - K_x \equiv 0 \tag{17}$$

3. Near Equivalence for the Type-3 PLL with the Classical Type-2 PLL

For a given value of ω_c, a Type-3 PLL behaves very similarly to a Type-2 PLL having a damping factor of ζ given by

$$\zeta \approx \frac{1}{2}\left(\frac{1-\alpha^2}{2\alpha}\right)^{\frac{1}{2}} \tag{18}$$

where $\alpha = (\omega_c\tau_2)^{-1}$

Assuming that we have a pre-specified value for ω_c, we need to know how to select τ_2. From (18), we obtain that

$$\alpha = \frac{-8\zeta^2 \pm \sqrt{64\zeta^4 + 4}}{2} \qquad (19)$$

In order to have $\alpha > 0$, it is straight forward to show that the only acceptable solution from (19) is

$$\tau_2^{-1} = \omega_c \left[\sqrt{16\zeta^4 + 1} - 4\zeta^2 \right] \qquad (20)$$

4. Phase Tracking Error for Type-3 PLL in the Presence of Orbit-Related Doppler

For the Type-3 PLL, the transfer function between the tracking phase error and the applied input phase function is given by

$$\frac{\theta_e(s)}{\theta_{in}(s)} = \frac{s^3}{s^3 + K_x(\tau_2 s)^2 + 2K_x\tau_2 s + K_x} \qquad (21)$$

and similarly for the output phase,

$$\frac{\theta_o(s)}{\theta_{in}(s)} = \frac{K_x(1+s\tau_2)^2}{s^3 + K_x(\tau_2 s)^2 + 2K_x\tau_2 s + K_x} \qquad (22)$$

For the phase error process, we let

$$\begin{aligned} u_0 &= \frac{d}{dt}\theta_{in} \\ u_1 &= \frac{d}{dt}\theta_e \\ u_2 &= \theta_e = \int u_1 dt \\ u_3 &= \int u_2 dt = \int \theta_e dt \\ u_4 &= \int u_3 dt = \iint \theta_e dt \end{aligned} \qquad (23)$$

which leads to the state-equations given by

$$\begin{aligned} u_0 &= 2\pi f_{in}(t) \\ u_1 &= u_0 - K_x\left[\tau_2^2 u_2 + 2\tau_2 u_3 + u_4\right] \\ u_2 &= \int u_1 dt \\ u_3 &= \int u_2 dt \\ u_4 &= \int u_3 dt \end{aligned} \qquad (24)$$

In making comparisons between the Type-2 and Type-3 systems, we should not equivalence ω_c with ω_n because the ω_c corresponds to approximately the unity open-loop gain for the Type-3 system whereas ω_n is the well-known radian natural loop frequency for a Type-2 PLL.

For the output phase of the loop when subjected to a given input phase trajectory, we have the transform relationship

$$\theta_o\left[s + K_x\tau_2^2 + \frac{2K_x\tau_2}{s} + \frac{K_x}{s^2} \right] = K_x\theta_{in}\left[\frac{1}{s^2} + \frac{2\tau_2}{s} + \tau_2^2 \right] \qquad (25)$$

The time behavior of the output phase can be computed by first making the following state variable assignments as

$$\begin{aligned} u_0 &= \theta_{in}(t) \\ u_1 &= \int u_0 dt = \int \theta_{in} dt \\ u_2 &= \int u_1 dt \\ u_3 &= \theta_o \\ u_4 &= \int u_3 dt \\ u_5 &= \int u_4 dt \end{aligned} \qquad (26)$$

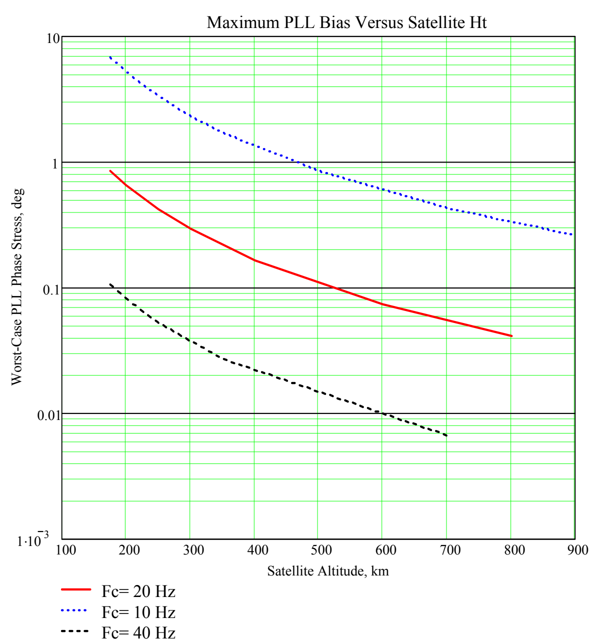
which leads to the state-variable equations that must be integrated as

$$
\begin{aligned}
u_0 &= \theta_{in}(t) \\
u_1 &= \int u_0 dt \\
u_2 &= \int u_1 dt \\
u_3 &= \theta_o(t) \\
u_4 &= \int u_3 dt \\
u_5 &= \int u_4 dt \\
\frac{d\theta_o}{dt} &= K_x\left[u_2 + 2\tau_2 u_1 + \tau_2^2 u_0\right] - K_x\left[\tau_2^2 u_3 + 2\tau_2 u_4 + u_5\right]
\end{aligned}
\tag{27}
$$

5. Type-3 PLL Tracking Error Results with Orbital Doppler Present

The preceding state equations were used along with the geometric orbit relationships to compute the PLL phase tracking error as a function of time and orbit altitude. The computations assumed that the RF carrier frequency was 2 GHz. The maximum phase error seen at the phase detector during a satellite pass is shown in Figure 6 as a function of PLL unity-gain bandwidth and orbital altitude (PLL gain margin constant at 20 dB).

Figure 6 Maximum PLL Bias Error Versus Satellite Altitude for Type-3 Tracking PLL

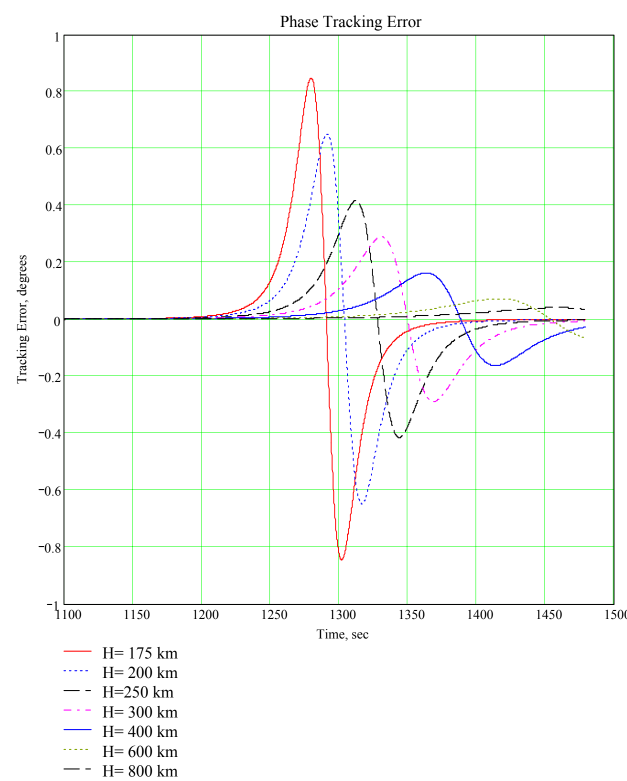


Due to the PLL's third-order characteristic equation, it should not be surprising that doubling the PLL's unity-gain bandwidth causes the worst-case phase error to be reduced by a factor of 2^3 approximately.

One specific set of computation results is shown in Figure 7 in order to illustrate the time-behavior of the tracking error during a satellite pass over the ground station. These results are based upon a PLL F_c= 20 Hz and a gain margin of 20 dB.

Reminder: None of the results presented here have included the effects of earth rotation or of elliptic orbits.

Figure 7 Phase Tracking Error for a Type-3 PLL (F_c= 20 Hz, G_M= 20 dB)



6. Accommodating Elliptical Non-Planar Orbits

To be continued…

7. References

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