Eigenfilter Design

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1 Introduction

A normal occurrence in modem design is the need for frequency-selective baseband filtering which does not introduce substantial intersymbol interference (ISI). The ISI introduced by this kind of filtering can be directly quantified from the time-sidelobe level of the filter's impulse response.

2 Problem Statement

The filter design problem must address three different design objectives:

1. Frequency passband should be flat to within a prescribed measure
2. Frequency stopband should be below a prescribed measure
3. Time-sidelobes of the impulse response should be below a prescribed level beyond a prescribed time duration from the main peak.

Some insight into the problem can be had by considering the requirements imposed by the third criterion above as follows here.

Assume that the filter's impulse response g(t) has G(f) as its Fourier transform. In order to minimize the energy content of the impulse response time sidelobes, we can choose to minimize the quantity

$$P_s = \int_{-\infty}^{+\infty} \left| g(t) \right|^2 dt - \int_{-Ts}^{+Ts} \left| g(t) \right|^2 dt$$

in which the main-lobe of the impulse response is to be limited to the temporal region $[-T_s, +T_s]$. The first integral represents all of the energy contained within the impulse response whereas the second integral represents the energy falling within the allowed main-lobe time region only. Making use of the Fourier inversion formula and substituting in the Fourier transform for g(t), this can be transformed to

$$P_s = \int_{-\infty}^{+\infty} df \int_{-\infty}^{+\infty} df' G(f) G^*(f') \frac{\sin\left[2\pi (f - f')T_s\right]}{\pi (f - f')T_s}$$

Rather than necessarily dealing with the continuous integrals, this result can be put into discrete form as

$$P_s = \left(2T_s \Delta F\right)^2 \sum_{k=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} G(f_k) G^*(f_m) \frac{\sin\left[2\pi (f_k - f_m)T_s\right]}{\left[2\pi (f_k - f_m)T_s\right]}$$

which can be further re-written as

$$P_s = \left[G(f_k)\right]^T \left[R_{k,m}\right] \left[G^*(f_m)\right]$$

with

$$R_{k,m} = \frac{\sin\left[2\pi (f_k - f_m)T_s\right]}{2\pi (f_k - f_m)T_s}$$

This form is very attractive because it captures the time sidelobe minimization problem as an eigensystem problem. The G(f) values that minimize P_s correspond to the eigenvector of [R] that has the minimum magnitude eigenvalue. Although this problem formulation does not capture the passband and stopband frequency response constraints that we also wish to impose on the design, it does lead us to consider the filter design problem from the eigenfilter perspective which is cast in a very similar form. The eigenfilter design method [1] is a very convenient yet optimal method for designing such filters.

3 Eigenfilter Design

The eigenfilter design problem can be formulated following the description provided in [1]. In this concept, assume that we are limiting ourselves to the design of a Type-1 FIR for which the impulse response g(n) = g(N-n) and N is an even number. The z-transform for this filter can be written as

$$H(z)=\sum_{n=0}^{N} g(n) z^{-n} = e^{-j\omega M}\sum_{n=0}^{M} b_n \cos(\omega n)$$
$$= e^{-j\omega M} H_R(\omega)$$

in which M= N/ 2. Adopting the notation

$$\underset{\to}{b} = [b_0, b_1, ..., b_M]^T$$
$$\underset{\to}{c}(\omega) = [1, \cos(\omega), \cos(2\omega), ..., \cos(m\omega)]^T$$

then

$$H_R^2(\omega) = \underset{\to}{b}^T \underset{\to}{c}(\omega)\underset{\to}{c}(\omega)^T \underset{\to}{b}$$

Energy content within the stopband can then be written as

$$E_s = \int_{\omega_s}^{+\infty} \left|H\left(e^{j\omega}\right)\right|^2 \frac{d\omega}{\pi}$$
$$= \underset{\to}{b}^T \int_{\omega_s}^{+\infty} \underset{\to}{c}(\omega)\underset{\to}{c}(\omega)^T \frac{d\omega}{\pi} \underset{\to}{b}$$
$$= \underset{\to}{b}^T P \underset{\to}{b}$$

The energy "error" in the passband can be similarly cast as

$$E_p = \underset{\to}{b}^T \int_0^{\omega_p} \left[1-\underset{\to}{c}(\omega)\right]\left[1-\underset{\to}{c}(\omega)\right]^T \frac{d\omega}{\pi}$$
$$= \underset{\to}{b}^T Q \underset{\to}{b}$$

The time sidelobes design constraint can be included within the problem by quantifying the amount of impulse response energy that occurs outside the main-lobe region as

$$E_t = \sum_{n=0}^{M} w(n) b_n^2$$

where w(n) is a weighting factor that can be chosen quite arbitrarily. This can be re-written in the same quadratic form as

$$E_t = \underset{\to}{b}^T S \underset{\to}{b}$$

in which matrix S is a diagonal matrix that contains the time sidelobe weighting factors like

$$S = \begin{bmatrix} w_0 & 0 & ... & 0 \\ 0 & w_1 & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & w_M \end{bmatrix}$$

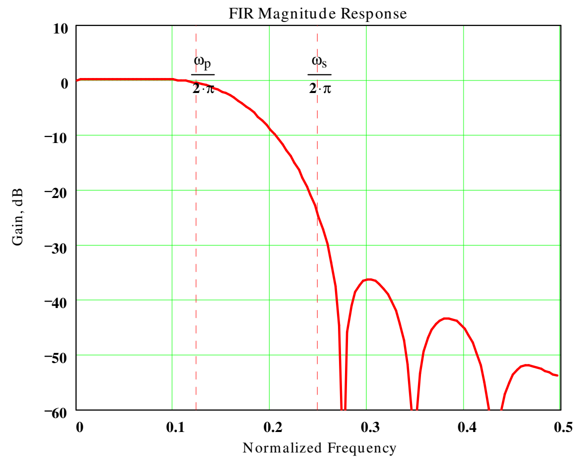
Putting these ingredients together, the objective function that we must minimize can be written directly as

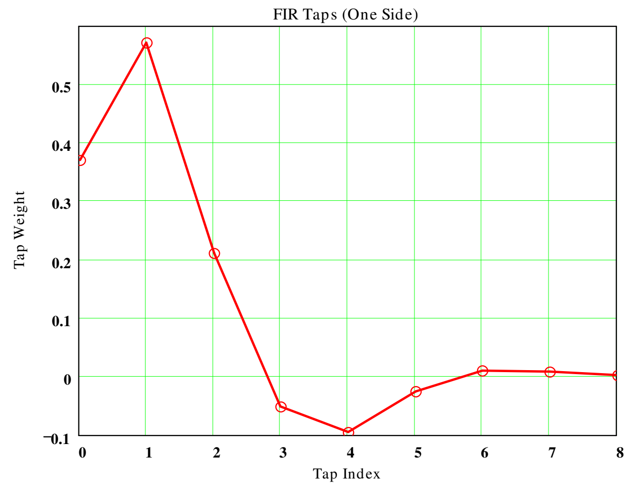
$$\Lambda = \alpha_s E_s + \alpha_p E_p + (1-\alpha_s-\alpha_p)E_t$$
$$= \underset{\to}{b}^T \left[\alpha_s P + \alpha_p Q + \left(1-\alpha_s-\alpha_p\right) S\right] \underset{\to}{b}$$

Recognizing that this is again a quadratic form and that the matrix represented by the bracketed quantity must be positive semi-definite, the minimization of Λ can be done by finding the eigenvector of the matrix having the smallest eigenvalue.

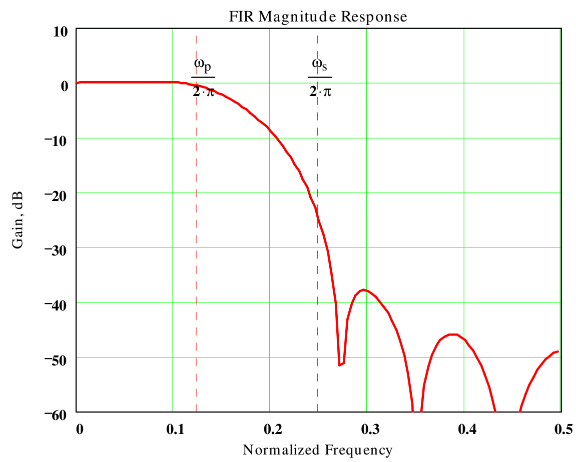
4 Design Examples

Several design examples can be helpful. In the cases that follow, the passband was designated as [0, F_s/ 8] and the stopband as [F_s/ 4, F_s/ 2] where F_s is the sampling rate involved. The weighting factor chosen for the time sidelobes was $(1.25)^{n-5}$ for FIR tap indices > 5, and α_p= 0.50, α_s= 0.35.


FIR Magnitude Response
$\frac{\omega_p}{2\cdot\pi}$
$\frac{\omega_s}{2\cdot\pi}$
10
0
−10
−20
−30
−40
−50
−60
Gain, dB
0
0.1
0.2
0.3
0.4
0.5
Normalized Frequency


FIR Taps (One Side)
0.5
0.4
0.3
0.2
0.1
0
−0.1
Tap Weight
0
1
2
3
4
5
6
7
8
Tap Index

N= 16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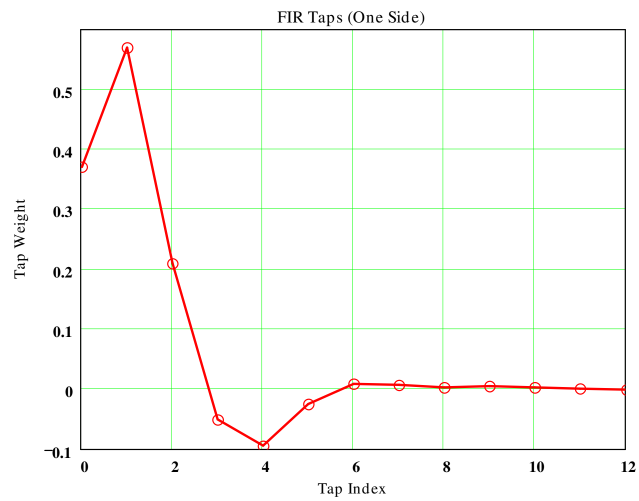
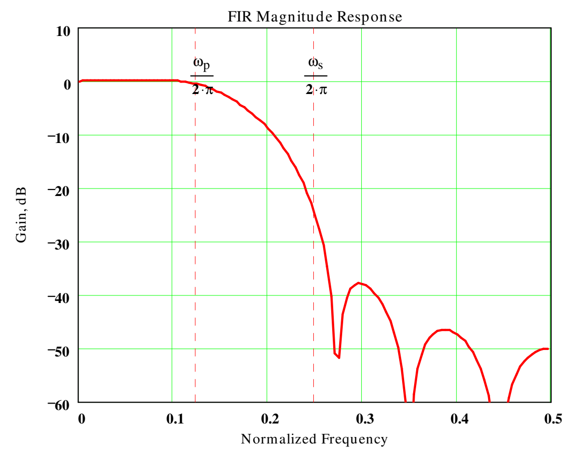
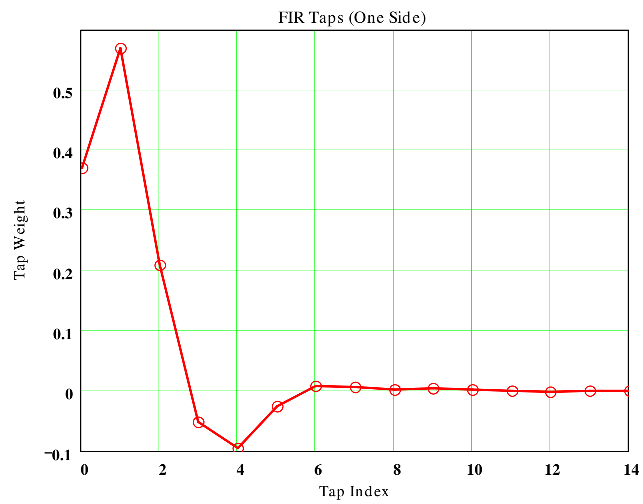
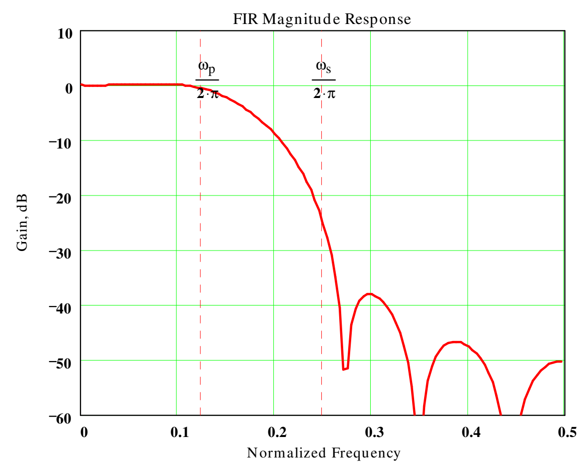

FIR Taps (One Side)
0.5
0.4
0.3
0.2
0.1
0
−0.1
Tap Weight
0
2
4
6
8
10
12
Tap Index

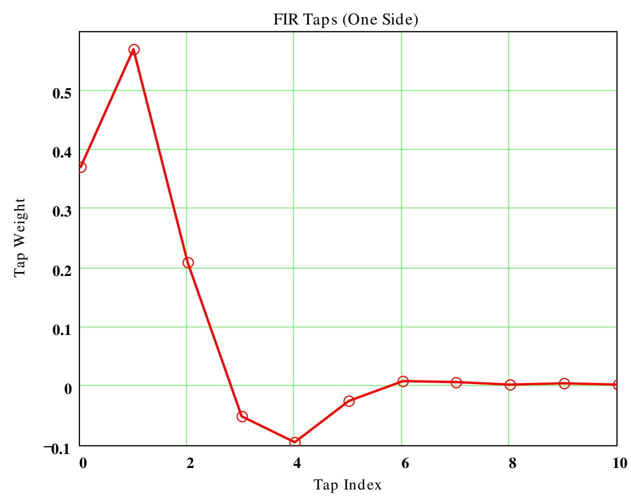
N= 24





N= 28





N= 48

5 References

1. Vaidyanathan, P.P., Multirate Systems and Filter Banks, Prentice-Hall, 1993

Eigenfilter Example

$N_{taps} := 16$ (must be an even number) $M := \frac{N_{taps}}{2}$

$\omega_p := \pi \cdot 0.25$

$\alpha_s := 0.35$ $\alpha_p := 0.5$

$\omega_s := \pi \cdot 0.5$

$rr := 0 .. M$ $\alpha_t := 1 - \alpha_s - \alpha_p$

$cc := 0 .. M$

$$P_{rr, cc} := \frac{1}{\pi} \cdot \int_{\omega_s}^{\pi} \cos(\omega \cdot rr) \cdot \cos(\omega \cdot cc)\, d\omega$$

$S_{rr, cc} := 0$

$$Q_{rr, cc} := \frac{1}{\pi} \cdot \int_{0}^{\omega_p} \left(1 - \cos(\omega \cdot rr)\right) \cdot \left(1 - \cos(\omega \cdot cc)\right) d\omega$$

$S_{rr, rr} := \mathrm{if}\left(rr > 5, 1.25^{rr-5}, 0\right)$

$R := \alpha_s \cdot P + \alpha_p \cdot Q + \alpha_t \cdot S$

$Evals := \mathrm{eigenvals}(R)$

$$Evals^T = \begin{pmatrix} 0.175 & 0.203 & 0.127 & 0.261 & 2.003 \times 10^{-4} & 0.013 & 0.359 & 0.425 & 1.227 \end{pmatrix}$$

$smallest := \min(Evals)$ $pick := \sum_{rr} \mathrm{if}\left(Evals_{rr} = smallest, rr, 0\right)$ $pick = 4$

$Vec := \mathrm{eigenvec}\left(R, Evals_{pick}\right)$

$$vv := \sum_{rr} Vec_{rr}$$

$$b := \frac{Vec}{vv}$$

$$b^T = \begin{pmatrix} 0.37 & 0.57 & 0.211 & -0.051 & -0.096 & -0.026 & 0.01 & 7.613 \times 10^{-3} & 3.279 \times 10^{-3} \end{pmatrix}$$

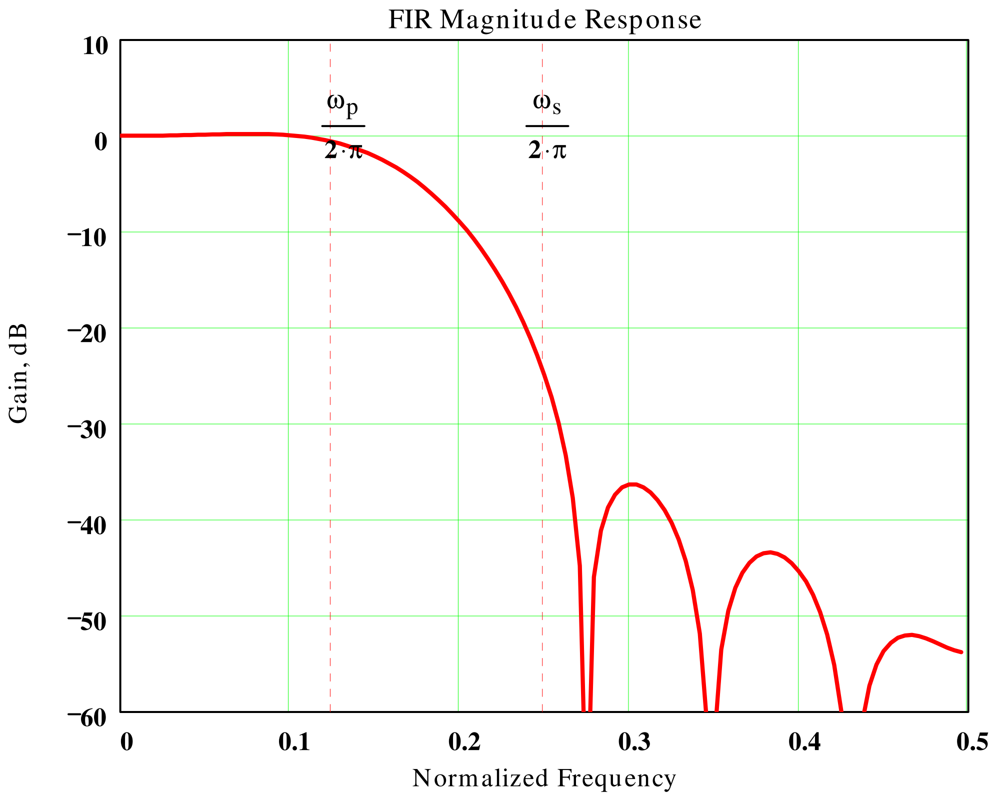
$$H_R(\omega) := \sum_{rr} b_{rr} \cdot \cos(rr \cdot \omega)$$

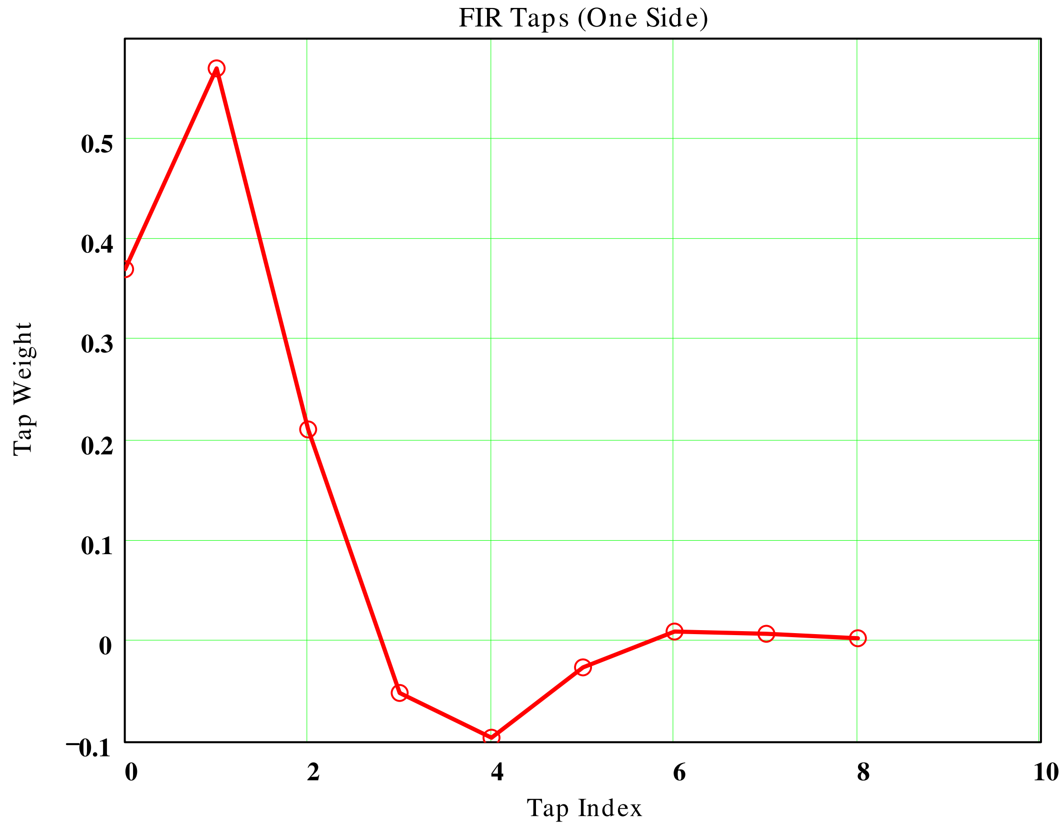
$$N_p := 120$$

$$pp := 0 .. N_p - 1$$

$$wsw_{pp} := \frac{pp}{N_p} \cdot \pi$$

$$mg_{pp} := 10 \cdot \log\left[\left(\left|H_R\left(wsw_{pp}\right)\right|\right)^2\right]$$

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