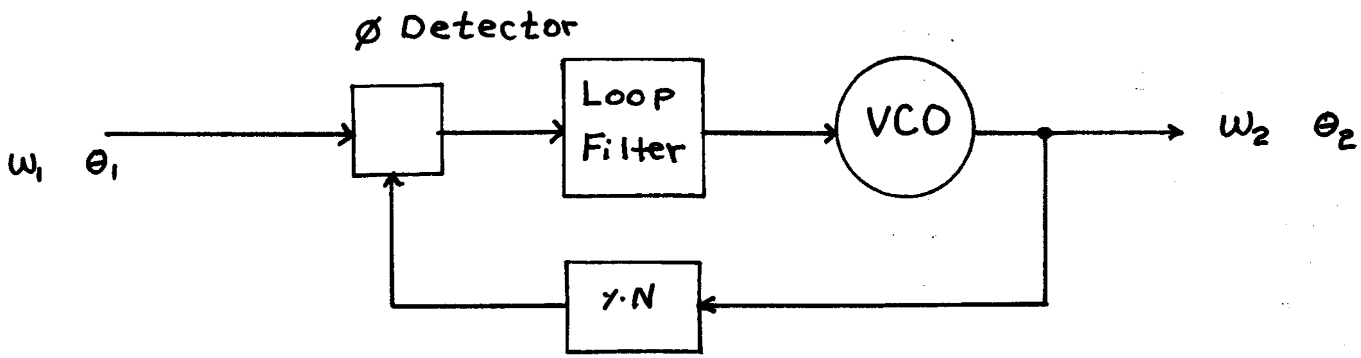
The Phase-Locked Loop Concept for Frequency Synthesis:

19 Feb. 1987

Time Domain Behavior



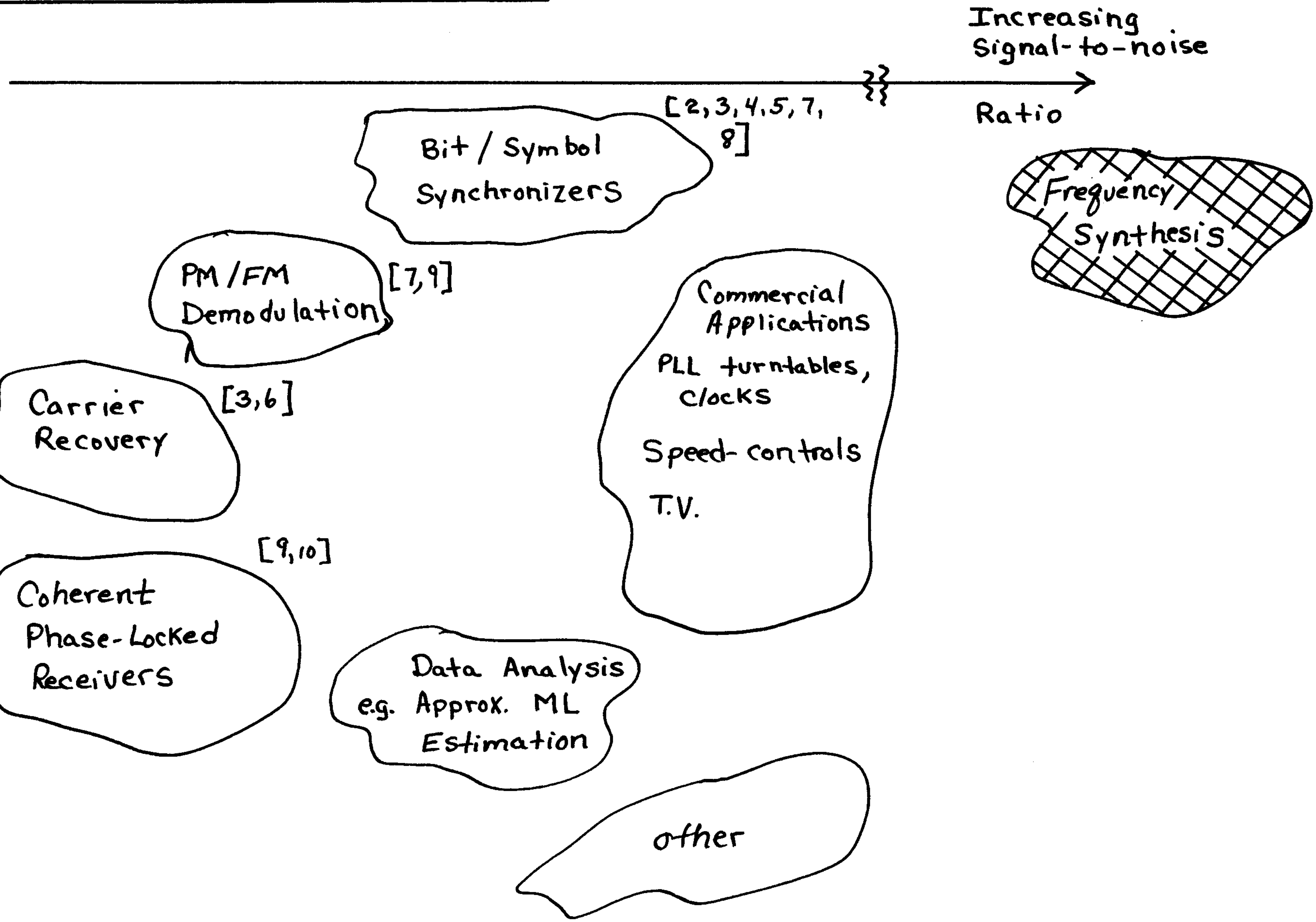
Background

The earliest description of phaselock was published by de Bellescize in 1932 and treated the synchronous reception of radio signals [1].

The first widespread use of phaselock was in the synchronization of horizontal and vertical scan in television receivers [1].

We will only consider a narrow avenue of phaselook applications.

Phase-Locked Loop Applications



Time Domain Performance Measures

Noise Measures: Wide-Sense Stationary Processes [11]

Wiener-Kinchin Theorem

$$S_\phi(\omega) = \int_{-\infty}^{\infty} R_\phi(\tau)\, e^{-j\omega\tau}\, d\tau$$

$$R_\phi(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_\phi(\omega)\, e^{j\omega\tau}\, d\omega$$

Variance of Average Frequency Departure

$$\sigma^2\left[\left\langle \dot{\phi} \right\rangle_{t,\tau}\right] = \frac{2}{\tau^2}\left[R_\phi(0) - R_\phi(\tau)\right]$$

$$= \frac{2}{\pi\tau^2}\int_{-\infty}^{\infty} S_\phi(\omega)\, \sin^2\left(\frac{\omega\tau}{2}\right) d\omega$$

Variance of Phase

$$\sigma^2\left[\phi(t)\right] = R_\phi(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_\phi(\omega)\, d\omega$$

Variance of Accumulated Phase

$$\sigma^2\left[\Delta_\tau \phi\right] = 2\left[R_\phi(0) - R_\phi(\tau)\right]$$

(≥ 0 because R_ϕ is positive semi-definite)

Noise Measures: Local Oscillator Phase Noise (App. E)

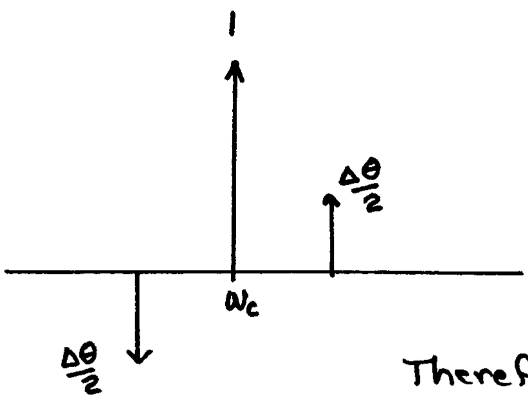
Consider a carrier with a noise component at offset frequency ω_m.

$$S(t) = \cos[\omega_c t + \underset{\uparrow \text{ peak phase deviation}}{\Delta\theta} \sin \omega_m t]$$

$$= \cos(\omega_c t)\cos(\Delta\theta \sin \omega_m t) - \sin(\omega_c t)\sin(\Delta\theta \sin \omega_m t)$$

For $\Delta\theta \ll 1$

$$S(t) \approx \cos(\omega_c t) - \frac{\Delta\theta}{2}\left[\cos(\omega_c - \omega_m)t - \cos(\omega_c + \omega_m)t\right]$$



By definition

$$\mathcal{L}(f_m) = \frac{\text{SSB Power @ } f_m}{\text{Carrier Power}} = \frac{\left(\frac{\Delta\theta}{2}\right)^2/2}{1/2}$$

$$= \frac{(\Delta\theta)^2}{4} = \frac{S_\phi(f_m)}{2} \qquad \text{Rad}^2/\text{Hz}$$

Therefore

$$\text{Variance}(\phi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_\phi(\omega)\,d\omega = \frac{1}{\pi}\int_{-\infty}^{\infty}\mathcal{L}(\omega)\,d\omega$$

Phase Variance due to synthesizer spectrum is found by integrating the $\mathcal{L}(f_m)$ spectrum.

For a finite observation time T

$$\sigma^2 = \int_{-\infty}^{\infty} S_\phi(\omega)\left[1 - \frac{\sin^2(\omega T/2)}{(\omega T/2)^2}\right]\frac{d\omega}{2\pi} \qquad \text{Rad}^2$$

Transient Response Measures

A system is said to have obtained phase-lock once specified conditions in the instantaneous phase and frequency errors (wrt. steady-state) have been obtained, e.g.

Settle to within ± 0.1 radian of final phase in $< 50\,\mu sec$

Some of the literature can be rather obscure, e.g. pertaining to stability and locking speed: [12]

> Definition 1 : Let G be a mapping of a vector space X onto itself, ie, $G: X \rightarrow X$. Then $x^* \in X$ is a fixed point of G if $G(x^*) = x^*$
>
> Definition 2 : Let X be a normed vector space and $G: X \rightarrow X$. If there exists L; $0 \le L < 1$ $\ni$
>
> $$\| G(x) - G(y) \| \le L \, \| x - y \|$$
>
> for all $x, y \in X$, then G is called a contractive mapping. (also called Lipschitz condition)

Theorem 1 Contraction Mapping Theorem

Let X be a complete normed vector space, and let $G : K \to K$ where K is a closed subset of X. Assume that the Lipschitz condition with $0 \le L < 1$ is satisfied for all $x, y \in K$. Then

1) there exists a unique fixed point $x^* \in K$ such that
$$G(x^*) = x^*$$

2) $\lim_{m \to \infty} x_m = x^*$ for any arbitrary choice of $x_0 \in K$

3) The rate of convergence is governed by
$$\| x_m - x^* \| \le \frac{L^m}{1-L} \| x_1 - x_0 \|$$

We will avoid obscurities where possible.

Phase-Locked Loop Transient Response

Continuous Systems

- Linear Loop Descriptions
  - Differential Equation
  - Laplace Transform
- Basic Response Types
- Complexities
  - Time Delays
  - Finite Gain-Bandwidth Op-amp
- General Analysis Approaches
  - Brute Force, Spice
  - Direct Inversion of Laplace Transform
  - State Variable / Transition Matrix Approach
  - Explicit Differential Equation
  - Other e.g. Corrington
- General Observations
  - Time Delay Effects
  - Post-Tuning Drift (See Appendix B)

Nonlinear Systems

- Describing Function or Harmonic Balance
- Phase-plane Analysis
- Stability

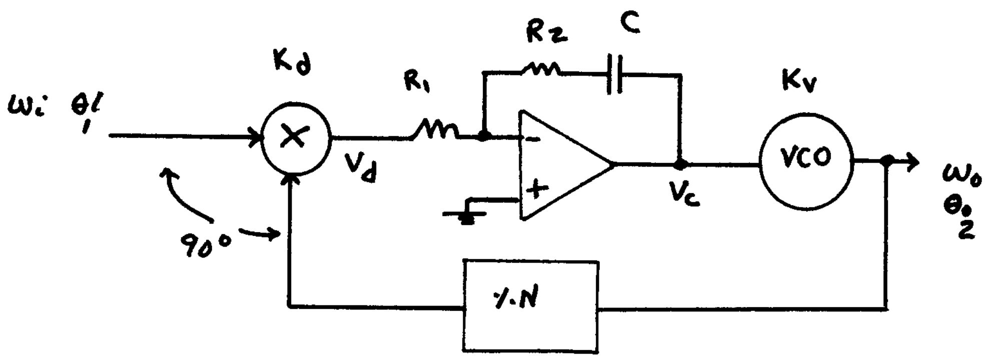
Sampled Systems

- When is Sampling a Consideration?
- General PLL Transfer Functions
- General Results for Type I & II Loops
- Miscellaneous Topics

Linear Phase-Locked Loop Descriptions

Consider a Type II system: Step Frequency & Phase Response

(Note: Loop type refers to the number of ideal integrators
Loop order refers to the order of the characteristic equation. [1])



Note: Inclusion of digital divider makes loop actually a sampled-control system, but this will be presently ignored.

K_d = Phase Detector Gain, V/Rad.

K_v = VCO sensitivity Rad./sec/V.

Standard Servo-System nomenclature:

Natural Frequency $\omega_n = \sqrt{\frac{K_d K_v}{N \tau_1}}$

Damping Factor $\zeta = \frac{1}{2} \omega_n \tau_2$

Time Constants $\tau_1 = R_1 C$

$\tau_2 = R_2 C$

The VCO acts as an ideal integrator :

$$\text{Total Output Phase} = \omega_c t + \int_0^t K_v V_c \, dt + \theta_2 = \theta_o$$

VCO center frequency

Assume $N=1$

Mixer - Phase Detector

Multiplying inputs together and dropping high frequency terms [13],

$$V_d = K_d \sin(\theta_1 - \theta_o)$$

$$= K_d \sin\left[(\omega_i - \omega_c)t - \theta_2 - \int_0^t K_v V_c(t) \, dt \right]$$

$$= K_d \sin \phi$$

where $\phi = \theta_i - \theta_o$

Lead- Lag Filter

Straight forward to show that

$$\tau_1 \dot{V}_c = -V_d - \tau_2 \dot{V}_d$$

Taking additional derivatives:

$$\dot{\phi} = \dot{\theta}_i - \dot{\theta}_o = \omega_i - \omega_c - K_v V_c$$

$$\ddot{\phi} = -K_v \dot{V}_c$$

From earlier

$$\dot{V}_c = \frac{-1}{\tau_1}\left[V_d + \tau_2 \dot{V}_d \right]$$

Substituting

$$\ddot{\phi} = -\frac{K_v}{\tau_1}\left[K_d \sin\phi + \tau_2 K_d \cos\phi \; \dot{\phi} \right]$$

or

$$\frac{d^2\phi}{dt^2} + \frac{K_d K_v}{\tau_1} \tau_2 \cos\phi \frac{d\phi}{dt} + \frac{K_d K_v}{\tau_1} \sin\phi \equiv 0$$

Note: Given an arbitrary memoryless phase detector characteristic $f(\cdot)$ which is everywhere differentiable, this result may be generalized to

$$\frac{d^2\phi}{dt^2} + \frac{K_d K_v}{\tau_1} \tau_2 \, f'(\phi) \frac{d\phi}{dt} + \frac{K_d K_v}{\tau_1} f(\phi) \equiv 0$$

or

$$\ddot{\phi} + 2 \xi \omega_n f'(\phi) \, \dot{\phi} + \omega_n^2 f(\phi) = 0$$

Linearizing assumption (or using a linear phase detector, not $\sin(\cdot)$)

$$f(\phi) = \phi$$

$$f'(\phi) \equiv 1 \qquad \text{Prime denotes derivative wrt } \phi$$

Substituting

$$\boxed{\ddot{\phi} + 2\zeta\omega_n \dot{\phi} + \omega_n^2 \equiv 1 \qquad \text{Linear Type II Diff. Eq.}}$$

Laplace Transform Description

The Laplace transform description could be obtained directly from the differential equation if desired.

Rather: Let θ_i = total input phase function

Lead-Lag transfer function	$\frac{1+s\tau_2}{s\tau_1}$
VCO transfer function	$\frac{K_v}{s}$
Phase Detector	K_d
Divide-by-N	$\frac{1}{N}$

Then $\quad V_d = K_d\left(\theta_i - \frac{\theta_o}{N}\right) = K_d\left[\theta_i - \frac{K_v}{sN} V_c(s)\right]$

$$V_d = K_d \left[\theta_i - \frac{K_v}{NS} \frac{1+s\tau_2}{S\tau_1} V_d \right]$$

$$V_d = \frac{K_d \theta_i}{1 + \frac{K_d K_v}{N\tau_1} \frac{1+s\tau_2}{S\tau_1}}$$

From this, the phase detector phase error is

$$\phi(s) = \frac{s^2 \theta_i(s)}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

where $\xi = \frac{1}{2}\omega_n \tau_2$ $\qquad \omega_n^2 = \frac{K_d K_v}{N\tau_1}$

From these results, it is straight forward to derive: [1]

0 dB open-loop BW $\qquad \omega_{0dB} = \omega_n \left[2\xi^2 + \sqrt{4\xi^4 + 1} \right]^{1/2}$

3 dB closed-loop BW $\qquad \omega_{3dB} = \omega_n \left[1 + 2\xi^2 + \sqrt{(2\xi^2+1)^2 + 1} \right]^{1/2}$

Phase Margin $\qquad \varphi_M = \tan^{-1}\left[\frac{2\omega_{0dB}\,\xi}{\omega_n} \right]$

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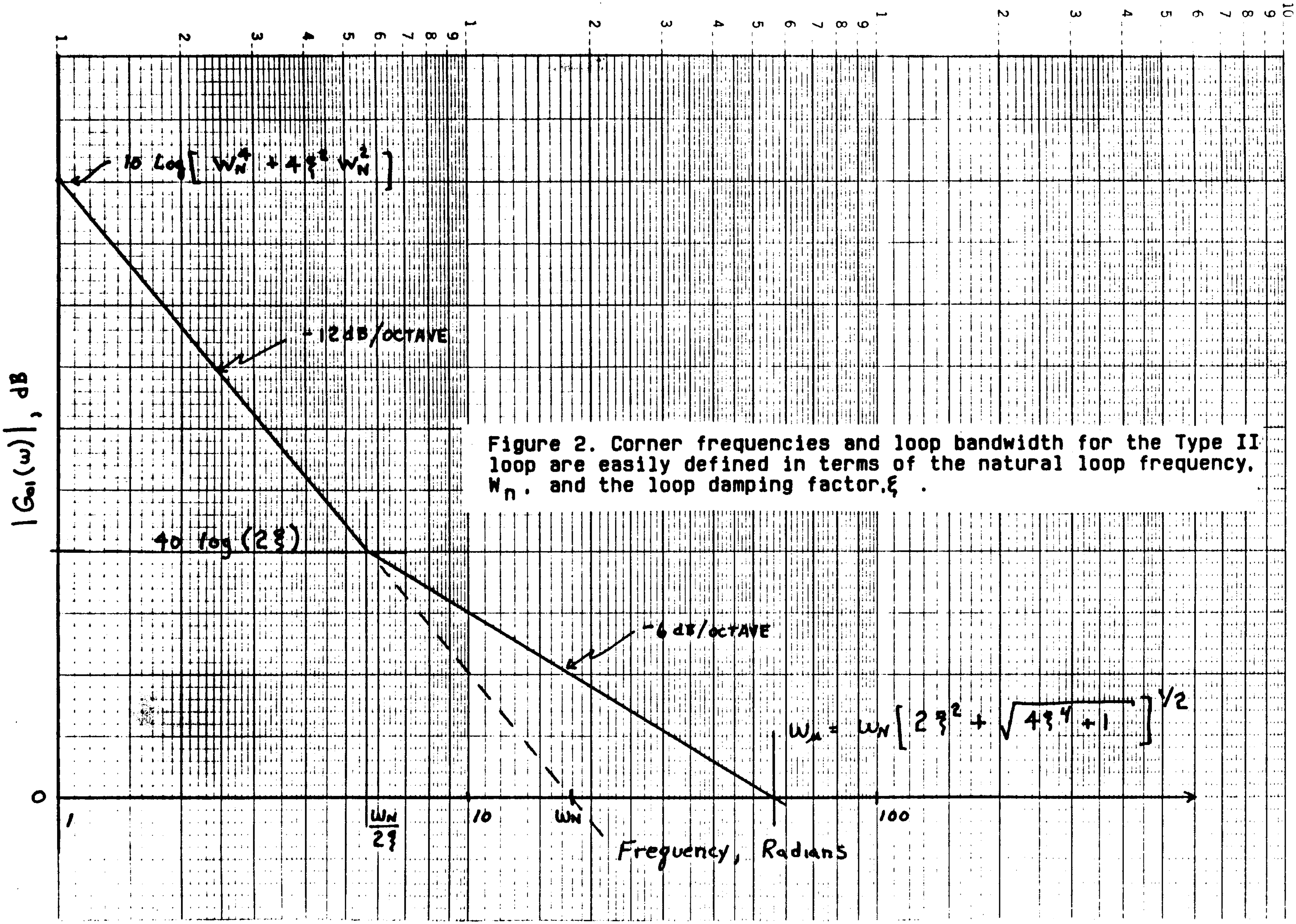


Figure 2. Corner frequencies and loop bandwidth for the Type II loop are easily defined in terms of the natural loop frequency, W_n, and the loop damping factor, ξ.

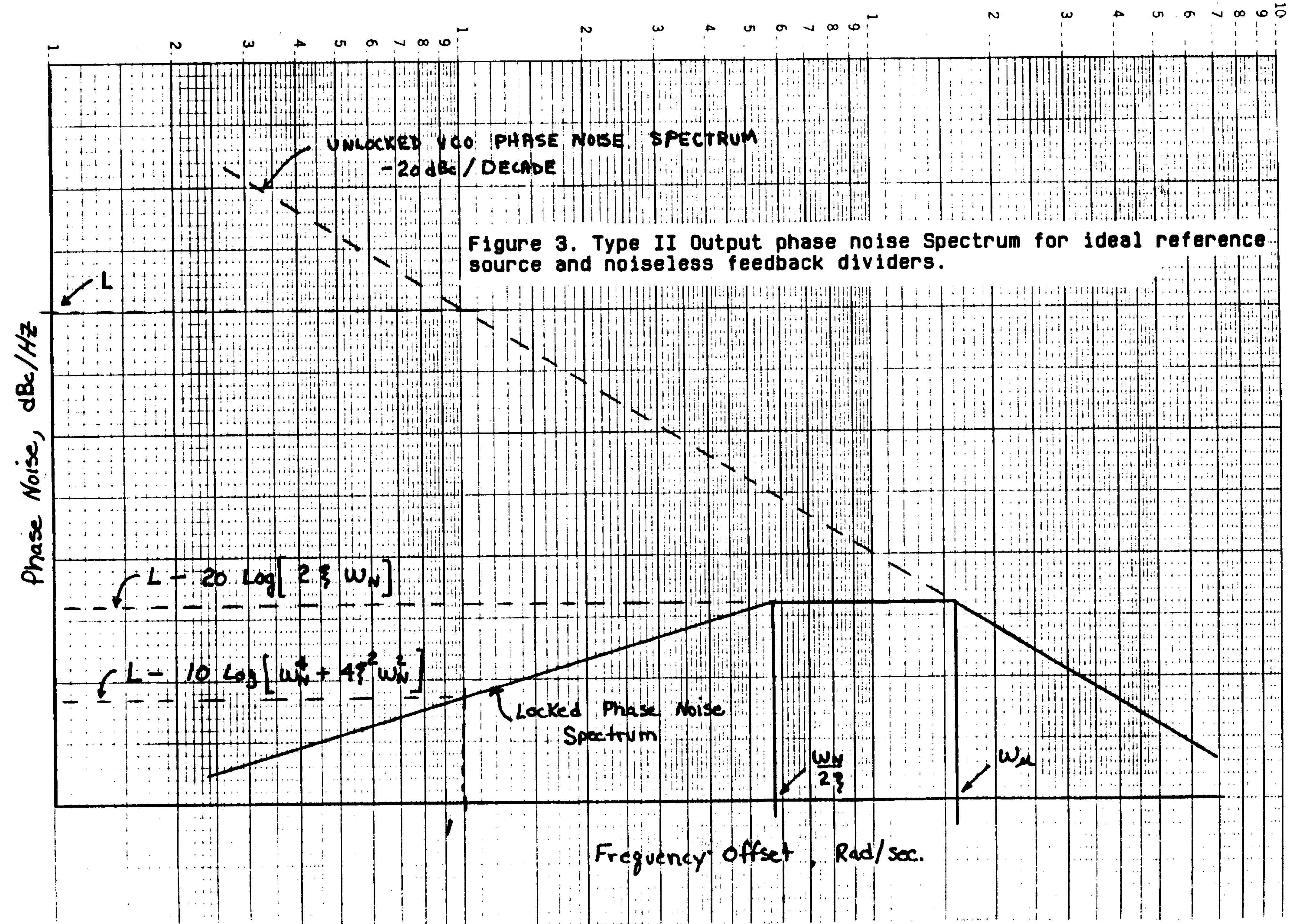


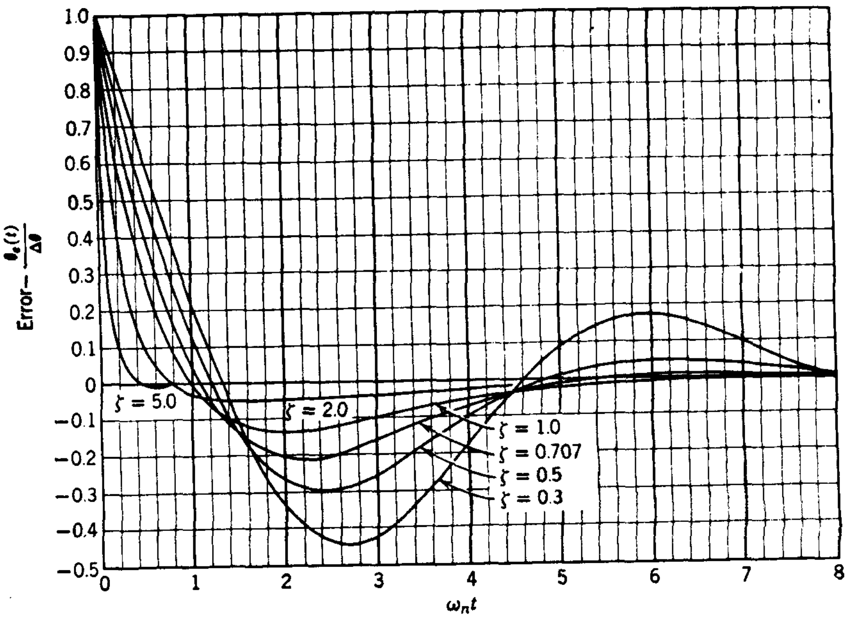
Figure 3. Type II Output phase noise Spectrum for ideal reference source and noiseless feedback dividers.

Transient Phase-Error Response [1]

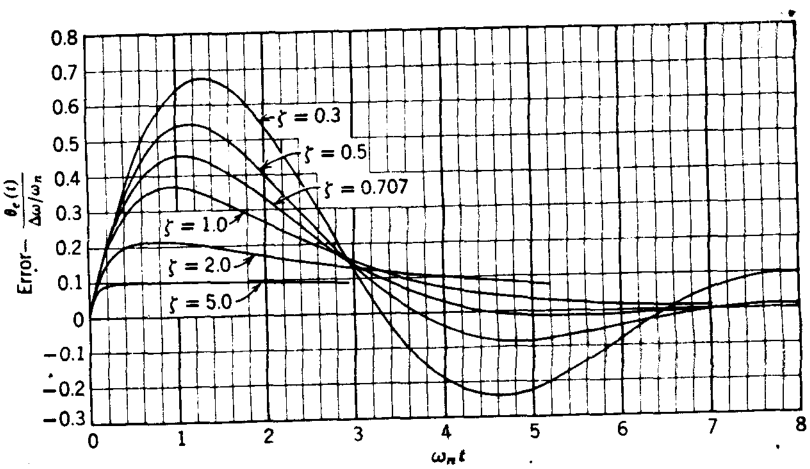
Transient Phase Error of Second-Order Loop, $\theta_e(t)$ (in rad) (high loop gain; $K_o K_d >> \omega_n$)

	Phase Step ($\Delta\theta$ rad)	Frequency Step ($\Delta\omega$ rad/sec)	Frequency Ramp ($\Delta\dot{\omega}$ rad/sec^2)
$\zeta < 1$	$\Delta\theta\left(\cos\sqrt{1-\zeta^2}\,\omega_n t - \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\sqrt{1-\zeta^2}\,\omega_n t\right)e^{-\zeta\omega_n t}$	$\frac{\Delta\omega}{\omega_n}\left(\frac{1}{\sqrt{1-\zeta^2}}\sin\sqrt{1-\zeta^2}\,\omega_n t\right)e^{-\zeta\omega_n t}$	$\frac{\Delta\dot{\omega}}{\omega_n^2} - \frac{\Delta\dot{\omega}}{\omega_n^2}\left(\cos\sqrt{1-\zeta^2}\,\omega_n t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\sqrt{1-\zeta^2}\,\omega_n t\right)e^{-\zeta\omega_n t}$
$\zeta = 1$	$\Delta\theta(1-\omega_n t)e^{-\omega_n t}$	$\frac{\Delta\omega}{\omega_n}(\omega_n t)e^{-\omega_n t}$	$\frac{\Delta\dot{\omega}}{\omega_n^2} - \frac{\Delta\dot{\omega}}{\omega_n^2}(1+\omega_n t)e^{-\omega_n t}$
$\zeta > 1$	$\Delta\theta\left(\cosh\sqrt{\zeta^2-1}\,\omega_n t - \frac{\zeta}{\sqrt{\zeta^2-1}}\sinh\sqrt{\zeta^2-1}\,\omega_n t\right)e^{-\zeta\omega_n t}$	$\frac{\Delta\omega}{\omega_n}\left(\frac{1}{\sqrt{\zeta^2-1}}\sinh\sqrt{\zeta^2-1}\,\omega_n t\right)e^{-\zeta\omega_n t}$	$\frac{\Delta\dot{\omega}}{\omega_n^2} - \frac{\Delta\dot{\omega}}{\omega_n^2}\left(\cosh\sqrt{\zeta^2-1}\,\omega_n t + \frac{\zeta}{\sqrt{\zeta^2-1}}\sinh\sqrt{\zeta^2-1}\,\omega_n t\right)e^{-\zeta\omega_n t}$
	Steady-state error $=0$	Steady-state error $= \frac{\Delta\omega}{K_v}$ (not included above)	Steady state error $= \frac{\Delta\dot{\omega}t}{K_v} + \frac{\Delta\dot{\omega}}{\omega_n^2}$ ($\Delta\dot{\omega}t/K_v$ not included above)

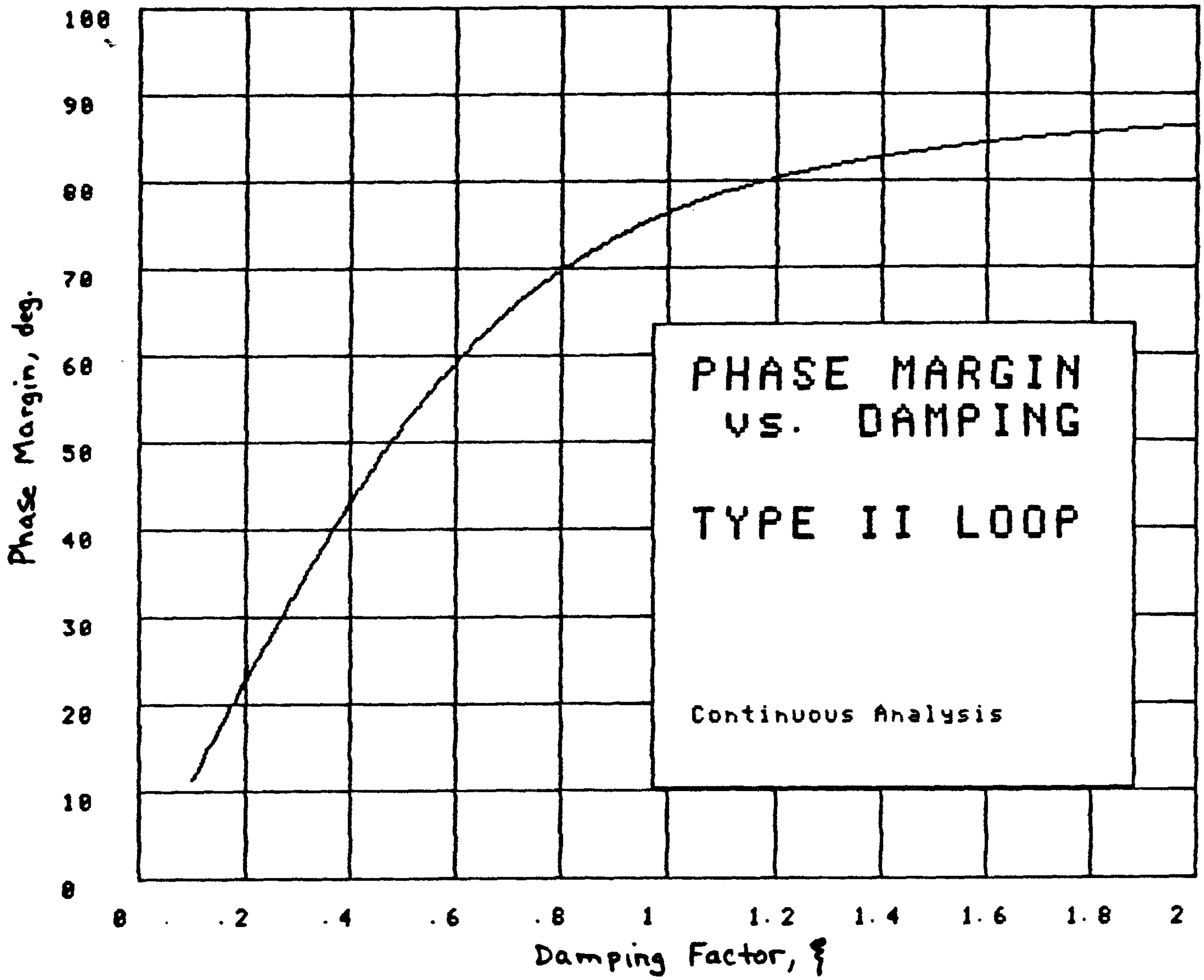
Transient Phase-Error Response [1]

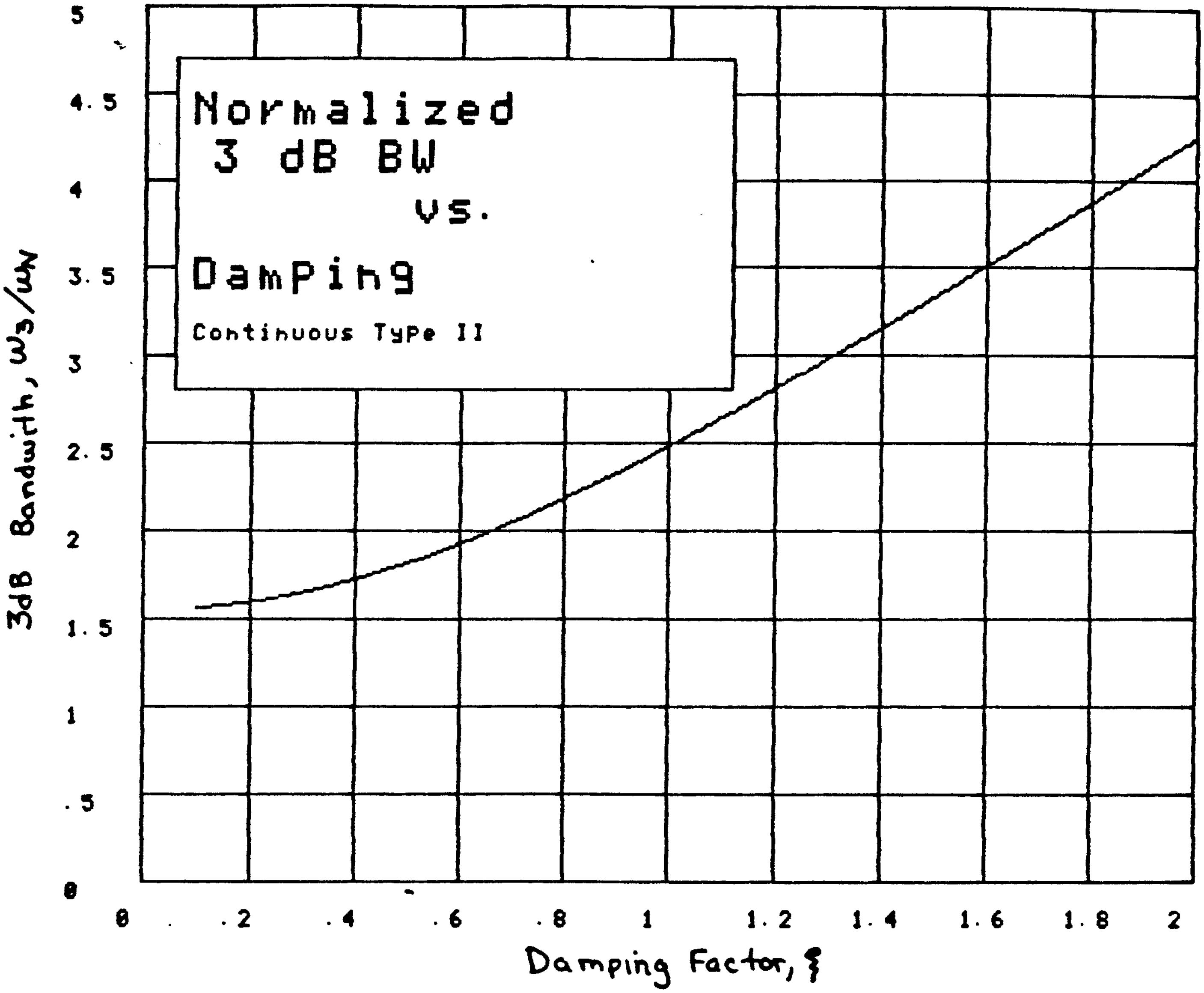


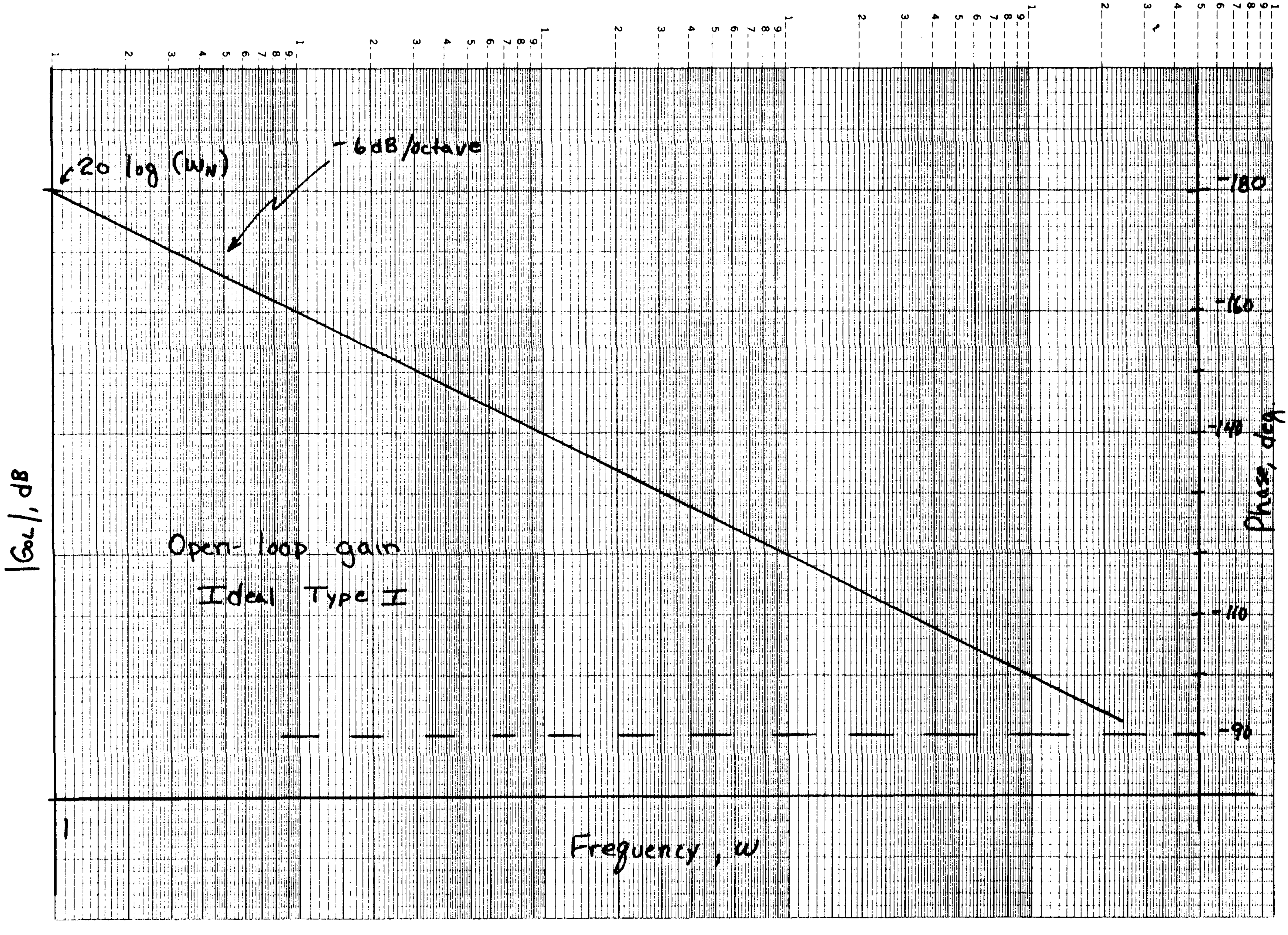
Phase error $\theta_e(t)$ due to a step in phase $\Delta\theta$. From Ref. 1 by permission of L. A. Hoffman.

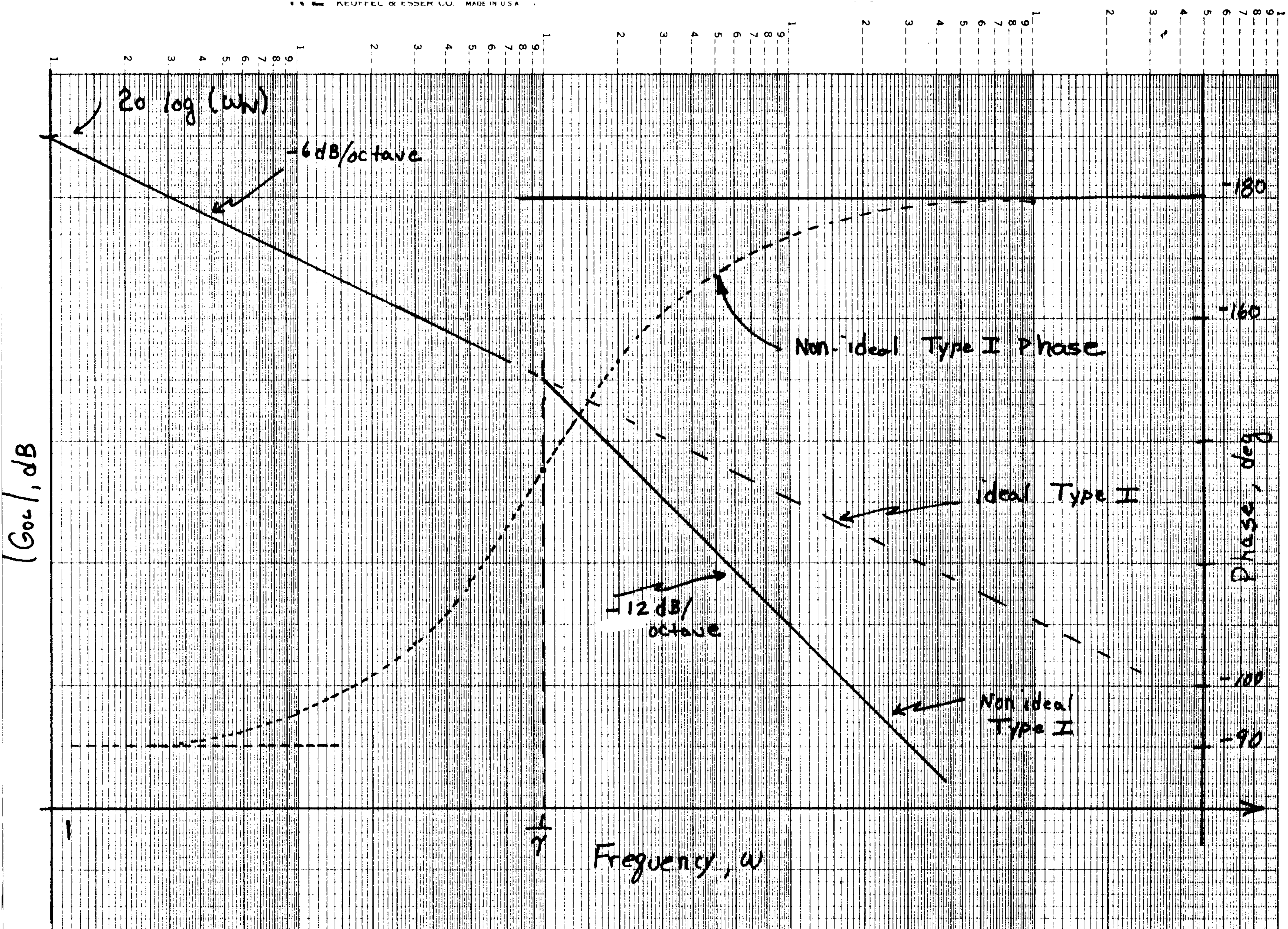


Transient phase error $\theta_e(t)$ due to a step in frequency $\Delta\omega$. (Steady-state velocity error, $\Delta\omega/K_v$, neglected.) From Ref. 1 by permission of L. A. Hoffman.

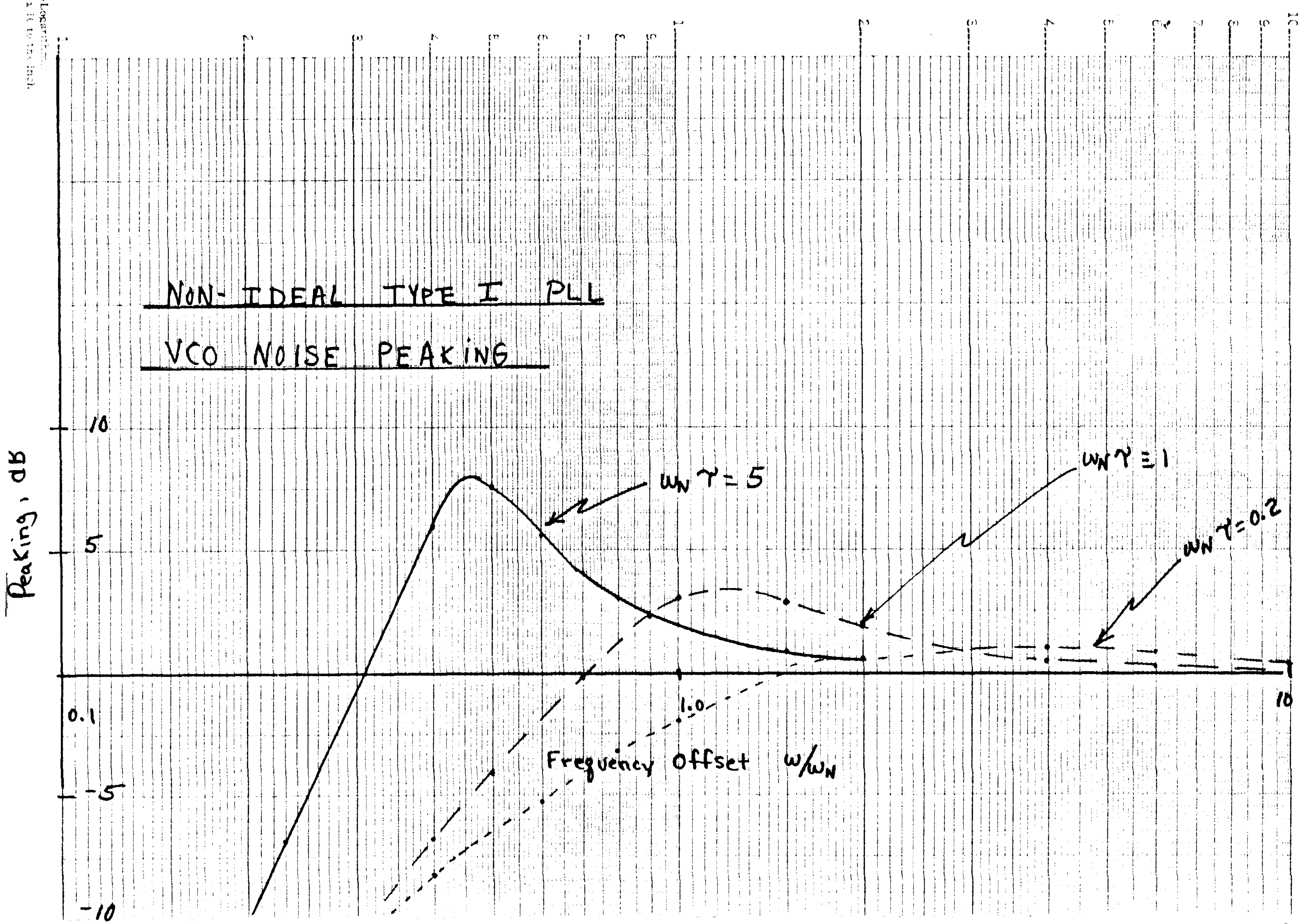

PHASE MARGIN vs. DAMPING
TYPE II LOOP
Continuous Analysis
Phase Margin, deg.
100
90
80
70
60
50
40
30
20
10
0
0
.2
.4
.6
.8
1
1.2
1.4
1.6
1.8
2
Damping Factor, ξ

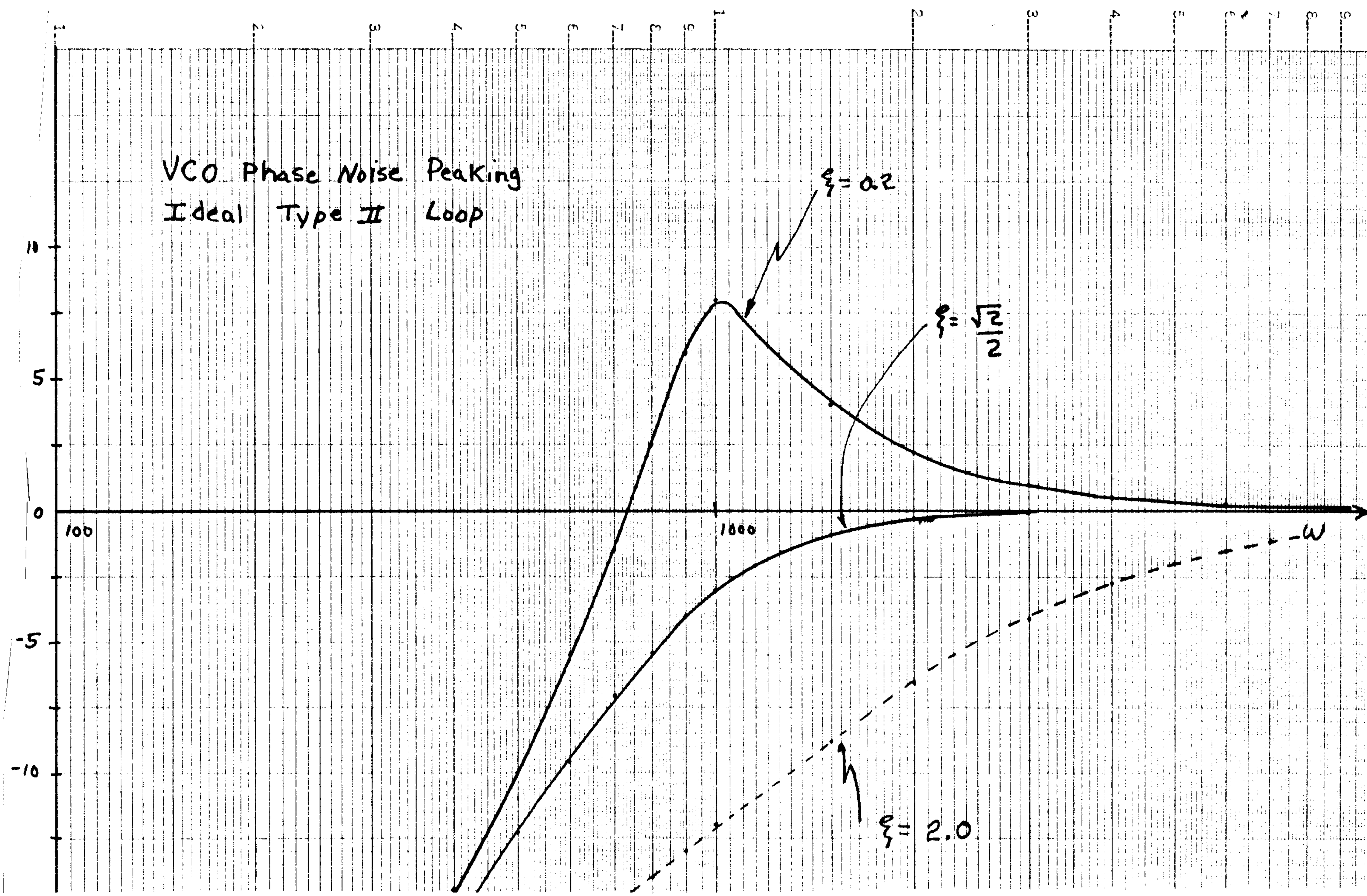

Normalized
3 dB BW
vs.
Damping
Continuous Type II
3dB Bandwith, ω_3/ω_N
5
4.5
4
3.5
3
2.5
2
1.5
1
.5
0
0
.2
.4
.6
.8
1
1.2
1.4
1.6
1.8
2
Damping Factor, ξ


20 log (ωN)
-6dB/octave
-180
-160
-140
-110
-90
Phase, deg
|GOL|, dB
Open-loop gain
Ideal Type I
1
Frequency, ω



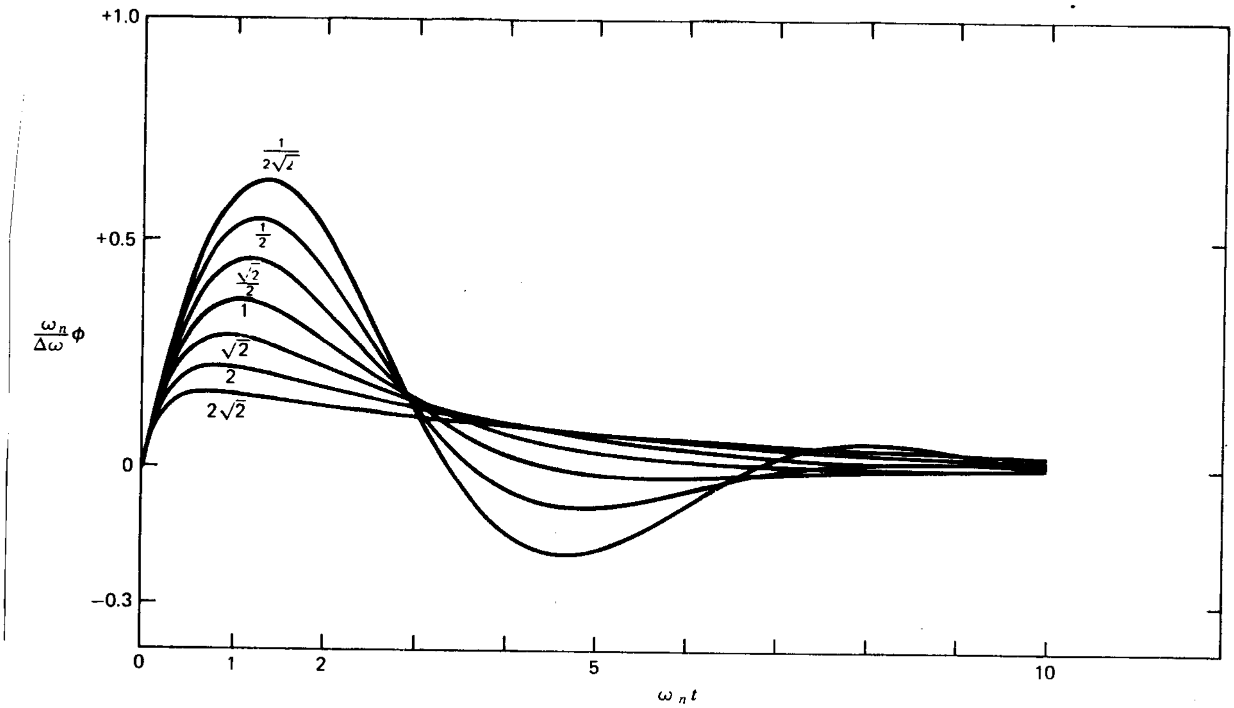
20 log (ωN)
-6 dB/octave
-180
-160
Non-ideal Type I Phase
ideal Type I
-12 dB/octave
Non ideal Type I
-100
-90
|Goc|, dB
Phase, deg
1
1/γ
Frequency, ω


NON-IDEAL TYPE I PLL
VCO NOISE PEAKING
Peaking, dB
10
5
-5
-10
0.1
1.0
10
$\omega_N \tau = 5$
$\omega_N \tau = 1$
$\omega_N \tau = 0.2$
Frequency Offset ω/ω_N



VCO Phase Noise Peaking
Ideal Type II Loop
ξ = 0.2
ξ = √2/2
ξ = 2.0
10
5
0
-5
-10
100
1000
ω

Frequency Step Response [14]



Phase error of a second-order loop $[F(s)=(1+\tau_2 s)/(\tau_1 s)]$ for a frequency step of the input signal.

Complexities: Internal Time Delay [15], [16]

Inclusion of true time delays within the control system complicates the analysis considerably.

Consider a Type I PLL:

Open-Loop Gain Function $G_{OL}(s) = \frac{\omega_n}{s} \underbrace{e^{-s\tau_d}}_{\text{Delay Term}}$

For a step-change in frequency, the exact phase-error solution is

$$\theta_E(t) = \frac{2\pi\,\Delta f}{N}\left[t + \sum_{n=1}^{\frac{t}{\tau_d}=N} (-\omega_n)^n \frac{(t-n\tau_d)^{n+1}}{(n+1)!}\,\mu(t-n\tau_d) \right]$$

In general, time delays are always detrimental to stability margins, switching speed, and phase noise peaking

For a Type II PLL:

$$G_{OL}(s) = \frac{\omega_n^2(1+2\zeta s/\omega_n)}{s^2}\, e^{-s\tau_d} \approx \frac{\omega_n^2(1+2\zeta s/\omega_n)}{s^2}\,\frac{1}{1+s\tau_d}$$

* No closed-form exact solution is possible for the Type II case.

Close approximations are possible.

Approximation of Time Delays in Continuous Systems

The Laplace transform of a pure time delay τ_d is

$$e^{-s\tau_d}$$

It is tempting to replace the exponential by a truncated Taylor series and continue on. However, this will in general introduce right-half plane poles which are really not present.

A technique which is finding increasing use in physical problems (but which is unsound in stability studies as well) is the use of the Padé approximant.

Padé Approximants for $e^{-s\tau}$

$n=1$

$$\frac{1 - s\tau/2}{1 + s\tau/2}$$

$n=2$

$$\frac{1 - s\tau/2 + (s\tau)^2/12}{1 + s\tau/2 + (s\tau)^2/12}$$

$n=3$

$$\frac{1 - s\tau/2 + (s\tau)^2/10 - (s\tau)^3/120}{1 + s\tau/2 + (s\tau)^2/10 + (s\tau)^3/120}$$

The Padé approximants may be further justified by noting that

$$e^{-s\tau} = \lim_{n\to\infty}\left(1 - \frac{s\tau}{n}\right)^n$$

$$e^{-s\tau} = \lim_{n\to\infty}\frac{\left(1 - \frac{s\tau}{2n}\right)^n}{\left(1 + \frac{s\tau}{2n}\right)^n}$$

As done in the Type II case, delays can be closely approximated by simply cascading RC lowpass filters.

Given $\dfrac{1}{1+s\tau_1}\;\dfrac{1}{1+s\tau_2}\;\dfrac{1}{1+s\tau_3} = H(s)$

$$\angle H(s) = -\tan^{-1}(\omega\tau_1) - \tan^{-1}(\omega\tau_2) - \tan^{-1}(\omega\tau_3)$$

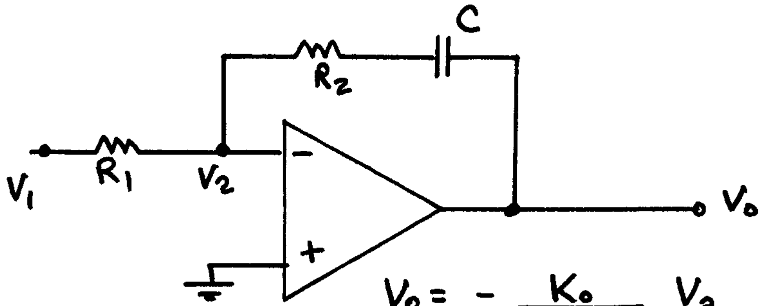
If $\dfrac{1}{2\pi\tau_i} \gg$ any frequency within the closed-loop bandwidth

$$\angle H(s) \approx -\sum_{i=1}^{N}\omega\tau_i$$

$$\tau \Rightarrow = -\frac{d\theta}{d\omega} = \sum_{i=1}^{N}\tau_i$$

Complexities: Finite Gain-Bandwidth Product, Lead-Lag Filter

Consider a finite Gain-Bandwidth Product lead-lag filter as given below:



$$V_o = -\frac{K_o}{1+s/\omega_o}\,V_2$$

$$-\frac{V_o}{V_1} = K_o \frac{1+s\tau_2}{s^2\,\frac{\tau_1+\tau_2}{\omega_o} + s\left[\frac{1}{\omega_o}+\tau_1+\tau_2+\tau_1 K_o\right]+1}$$

$$= \frac{K_o\omega_o}{\tau_1+\tau_2}\;\frac{1+s\tau_2}{s^2 + s\left(\frac{1/\omega_o+\tau_1+\tau_2+\tau_1 K_o}{(\tau_1+\tau_2)}\right)\omega_o + \frac{\omega_o}{\tau_1+\tau_2}}$$

In standard nomenclature

$$\omega_n^2 = \frac{\omega_o}{\tau_1+\tau_2}$$

$$\zeta = \frac{\frac{1}{\omega_o}+\tau_1+\tau_2+\tau_1 K_o}{2}\,\omega_n \approx \frac{K_o\tau_1}{2}\,\omega_n$$

Solving for denominator roots:

$$(s-a)(s-b) = s^2 - (a+b)s + ab$$

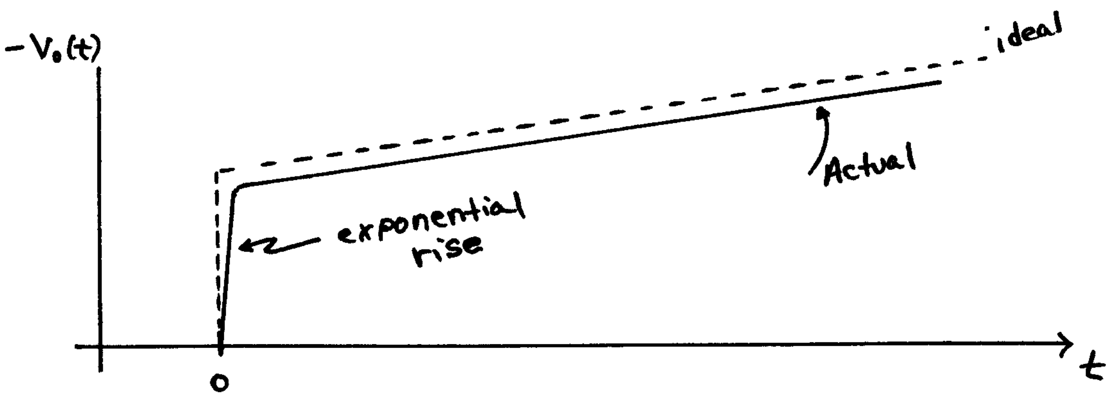
$\uparrow$ |larger root| dominates

$$\text{Root \#1} \doteq -2\zeta\omega_n = -K_o\tau_1\omega_n^2 = -\frac{K_o\omega_o}{1+\tau_2/\tau_1} = a$$

$$\text{Root \#2} \doteq -\frac{1}{\tau_1}$$

Evaluating $V_o(t)$ for a unit-step input:

$$V_o(t) = \frac{-K_o\omega_o}{\tau_1+\tau_2}\left\{\frac{\tau_2}{a-b}\left(e^{at}-e^{bt}\right) + \frac{1}{a-b}\left(\frac{e^{at}}{a} - \frac{e^{bt}}{b} - \frac{1}{a} + \frac{1}{b}\right)\right\}$$



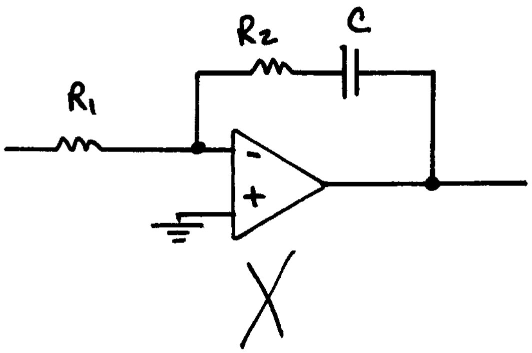
Avoiding Finite Gain-BW Problems

Note

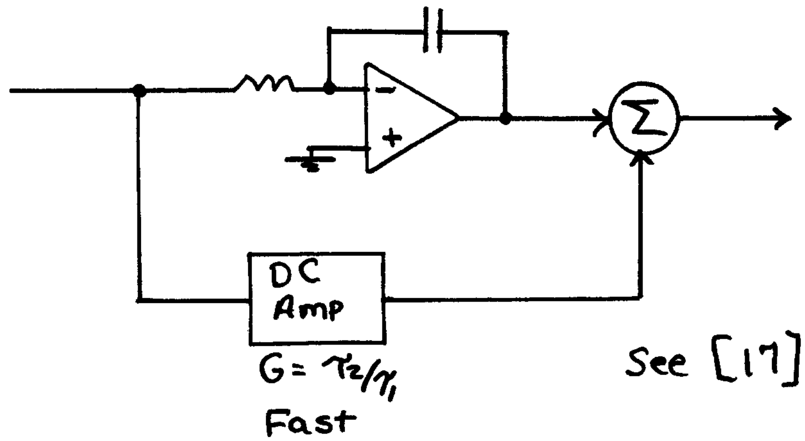
$$\frac{1+s\tau_2}{s\tau_1} = \frac{1}{s\tau_1} + \frac{\tau_2}{\tau_1}$$

Integrator

DC-Coupled Amplifier



Rather



See [17]

General Analysis Approaches

Brute Force Calculation

Programs such as spice can be used to calculate the transient response of a PLL.

e.g. model the VCO (ideal integrator) as a shunt capacitor.

For simple cases, hand calculation is possible, and for a novice, this can provide insight.

Note: Programs such as Spice generally use implicit integration rather than explicit. As a result, simple explicit solution for the system state variables and integration using std. routines (Runge-Kutta, Adams-Molton, Adams-Bashforth, etc.) will in general not be as stable. See [18] or [19] for details.

See Appendix A for an example where R_1 in the lead-lag filter is shunted by two diodes.

Direct Laplace Transform Inversion

Given the Laplace transform description of the phase error response, it is possible to directly evaluate the time domain response using residues.

The operation is two-step:

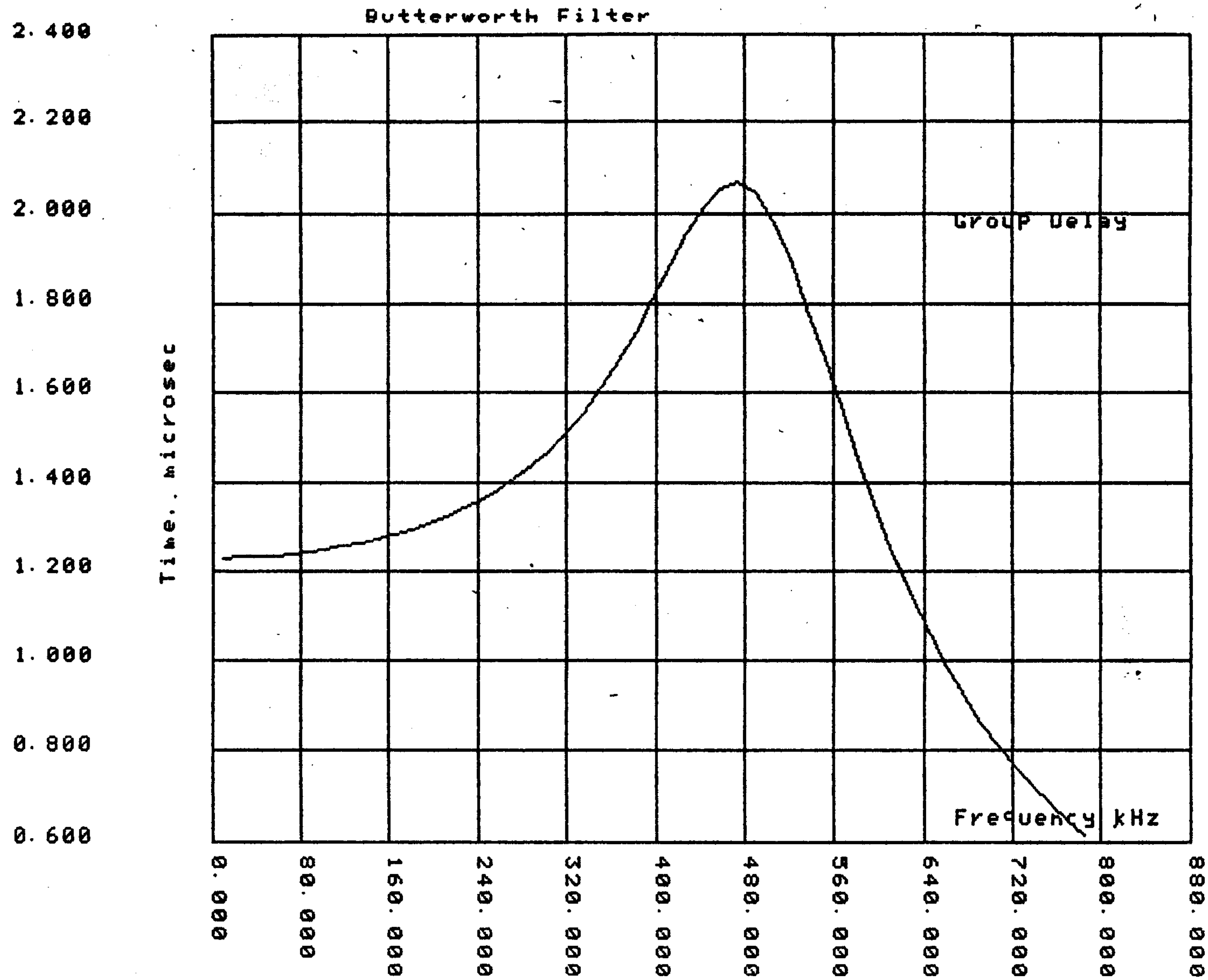
1) Calculate all denominator roots
2) perform a partial fraction expansion of the transform from which the time domain functions can be found from tables.

This approach is prone to instabilities in systems of order greater than 8 or 10.

This is particularly true when the describing differential equation is very "stiff" ie the roots are widely spread.

Techniques discribed in Cuthbert are nonetheless improved and good for this approach [20]

i) Moore Root Finder (makes use of Cauchy-Riemann eqns.)
ii) improved partial fraction expansion incl. multiple roots



Filter Characteristics:

Butterworth Filter

Filter Order = 6
Pass band frequency = 5.0000000000E+05
max. loss = 3.000
Stop band frequency = 1.0000000000E+06
min. loss = 35.000

Normalized Poles

Real Part	Imag. Part
-2.5881904514E-01	9.6592582628E-01
-7.0710678126E-01	7.0710678112E-01
-9.6592582633E-01	2.5881904494E-01
-9.6592582623E-01	-2.5881904533E-01
-7.0710678097E-01	-7.0710678140E-01
-2.5881904474E-01	-9.6592582638E-01

This is the filter "prelude" to my inter-Symbol interference analysis program. Program will work for any filter of interest. It is written in Turbo Pascal. If you'd like a copy, just say so. J.C.

CP/M-86 1.1 (1.107:015;A)   CAP ALT GR1 GR2   sp:9   Thu   05/30/85 21:13

Filter Order = 6

Factor = 9.9999999999E-01

The pole residues in input order

i	Re(Root)	Im(Root)	Re(Residue)	Im(Residue)
1	-2.5881904514E-01	9.6592582628E-01	2.0412414538E-01	3.5355339029E-01
2	-7.0710678126E-01	7.0710678112E-01	-1.5236033613E+00	1.2381254538E-09
3	-9.6592582633E-01	2.5881904494E-01	1.3194792140E+00	-2.2854050431E+00
4	-9.6592582623E-01	-2.5881904533E-01	1.3194792185E+00	2.2854050404E+00
5	-7.0710678097E-01	-7.0710678140E-01	-1.5236033613E+00	1.8558228280E-09
6	-2.5881904474E-01	-9.6592582638E-01	2.0412414465E-01	-3.5355339069E-01

Time Response

0.4082 exp(-0.2588 t) Cos(0.9659 t)

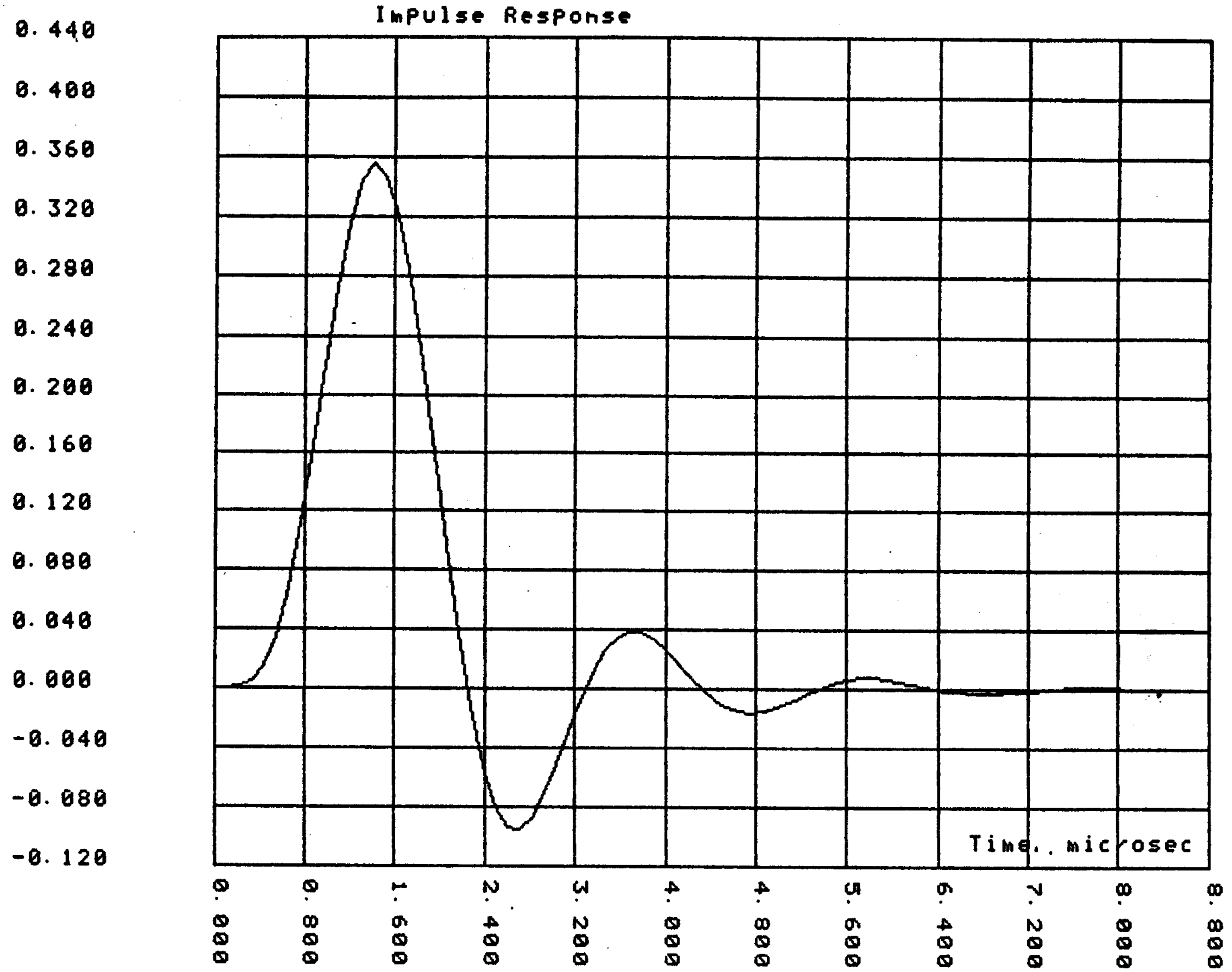
-0.7071 exp(-0.2588 t) Sin(0.9659 t)

-3.0472 exp(-0.7071 t) Cos(0.7071 t)

2.6390 exp(-0.9659 t) Cos(0.2588 t)

4.5708 exp(-0.9659 t) Sin(0.2588 t)

Impulse Response



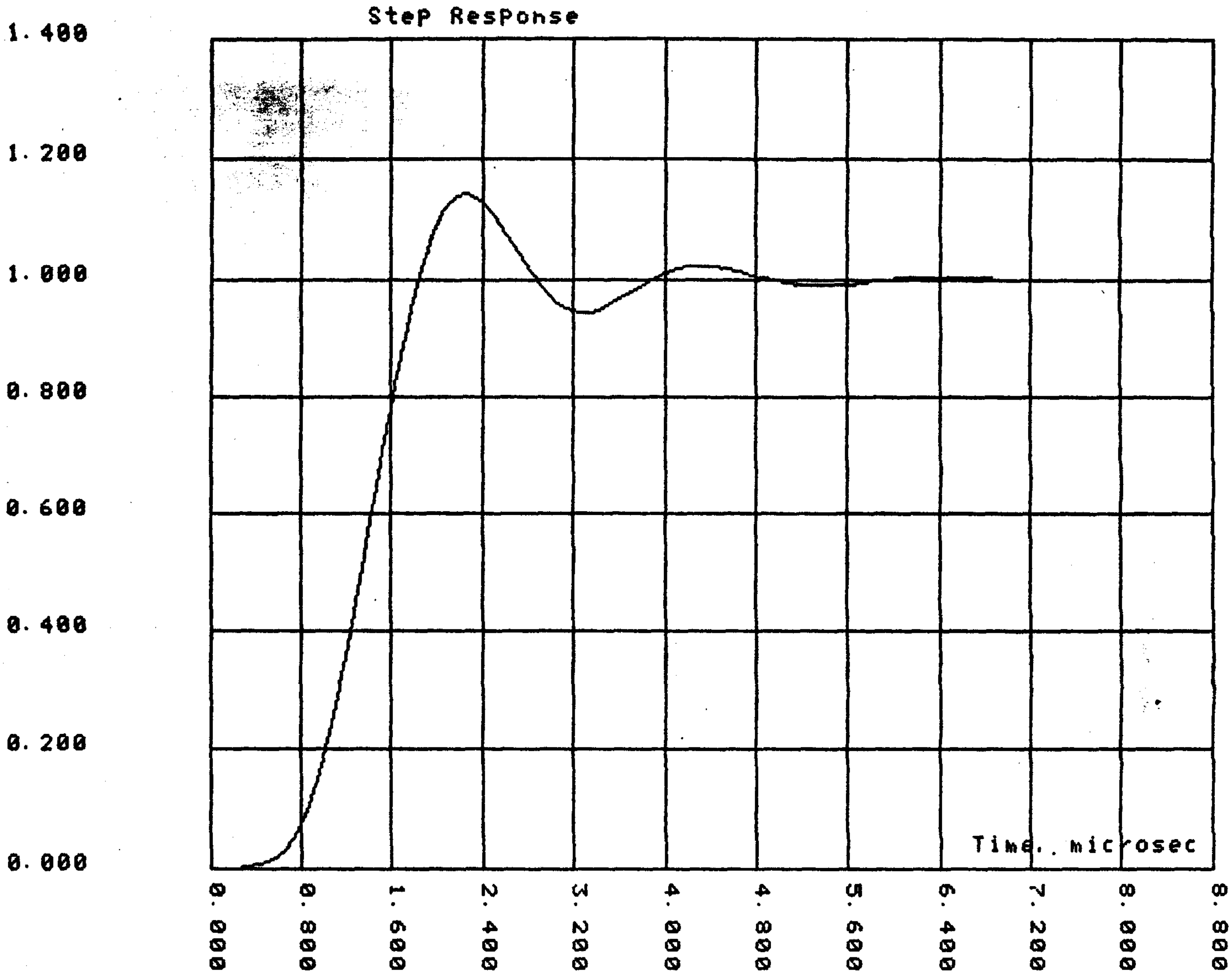
Filter Characteristics:

Butterworth Filter

Filter Order = 6
Pass band frequency = 5.0000000000E+05
max. loss = 3.000
Stop band frequency = 1.0000000000E+06
min. loss = 35.000

Normalized Poles

Real Part	Imag. Part
-2.5881904514E-01	9.6592582628E-01
-7.0710678126E-01	7.0710678112E-01
-9.6592582633E-01	2.5881904494E-01
-9.6592582623E-01	-2.5881904533E-01
-7.0710678097E-01	-7.0710678140E-01
-2.5881904474E-01	-9.6592582638E-01



Filter Characteristics:

Butterworth Filter

Filter Order = 6
Pass band frequency = 5.0000000000E+05
max. loss = 3.000
Stop band frequency = 1.0000000000E+06
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Normalized Poles

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-9.6592582623E-01	-2.5881904533E-01
-7.0710678097E-01	-7.0710678140E-01
-2.5881904474E-01	-9.6592582638E-01

State Variable / Transition Matrix Method

Given a system which is linear, but has time-varying elements, the dynamic equations take the form

$$\underline{\dot{x}}(t) = A(t)\,\underline{x}(t) + B(t)\,\underline{u}(t)$$

$$\underline{c}(t) = D(t)\,\underline{x}(t) + E(t)\,\underline{u}(t)$$

The state-transition matrix, $\phi(t,t_0)$ is defined as the $n \times n$ matrix which satisfies the homogeneous state equation

$$\underline{\dot{x}}(t) = A(t)\,\underline{x}(t)$$

The Solution is $\underline{x}(t) = \phi(t,t_0)\,\underline{x}(t_0)$

In the time-invariant case, the state-transition matrix is independent of t_0 and

$$\phi(t) = e^{At} = \sum_{k=0}^{\infty} A^k \frac{t^k}{k!}$$

↑ Caley-Hamilton theorem can be used to simplify matrix multiplications

Alternately, taking Laplace transforms of both sides of the homogeneous equation

$$\dot{\underline{X}} = A\,\underline{X} \quad \Rightarrow \quad s\,\underline{X}(s) - \underline{x}(0^+) = A\,\underline{X}(s)$$

$$\underline{X}(s) = (sI - A)^{-1}\,\underline{x}(0^+)$$

$$\therefore \ \phi(t) = \mathcal{L}^{-1}\left[(sI - A)^{-1}\right]$$

Using this solution, for the time-invariant case

$$\underline{x}(t) = \phi(t - t_0)\,\underline{x}(t_0) + \int_{t_0}^{t} \phi(t-\tau)\,B(\tau)\,\underline{u}(\tau)\,d\tau$$

See [21] or [22] for more details.

Explicit Differential Equation

Assume that the following rational function in s represents the Laplace transform of the phase error response to an arbitrary input transient.

$$X(s) = \frac{a_{m-1} s^{m-1} + a_{m-2} s^{m-2} + \cdots + a_1 s + a_0}{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}$$

It can be shown that this is equivalent to a specific differential equation in the time domain with specific initial conditions

The initial conditions are:

$$x^{(0)}(0^+) = \frac{a_{m-1}}{b_m}$$

indicates i^{th} order derivative

$$x^{(i)}(0^+) = \frac{1}{b_m}\left[a_{m-i-1} - \sum_{j=0}^{i-1} b_{m-i+j}\, x^{(j)}(0^+) \right] \qquad 1 \le i \le m-1$$

Additional derivatives may be calculated as

$$x^{(m+i)}(0^+) = -\frac{1}{b_m} \sum_{j=0}^{m-1} b_j\, x^{(j+i)}(0^+) \qquad \text{for } i \geq 0$$

Given the initial conditions of x and at least its first m derivatives, the time domain response may be calculated iteratively using a truncated Taylor series and updating each derivative value at each time increment.

$$X(t+T) = X(t) + \sum_{j=1}^{m+V} X^{(j)}(t) \frac{T^j}{j!}$$

V = # of additional derivatives more than m

Update derivatives

$$X^{(i)}(t+T) = \sum_{j=0}^{m+V-i} X(t)^{(i+j)} \frac{(T)^j}{j!}$$

Method is prone to round-off error problems which can lead to algorithm instability.

* * * * * * * * * * * * * *

I N V E R S E   L A P L A C E   T R A N S F O R M

R O S S   M E T H O D

Maximum order of polynomial = 3
Number of additional derivatives to be included 0
Time interval   .01
Number of timd interval points to calculate 30

INPUT POLYNOMIAL COEFFICIENTS IN PAIRS

Numerator S^0, Denominator S^0,   etc.

8	6
3	11
1	6
0	1

$$\frac{S^2+3S+8}{S^3+6S^2+11S+6} \Rightarrow 3e^{-t}-6e^{-2t}+4e^{-3t}$$

		Exact
.01	.9707396	.9707396
.02	.942916	
.03	.9164665	
.04	.891328	
.05	.8674376	
.06	.8447321	
.07	.8231486	
.08	.8026241	
8.999999E-02		.7830956
9.999999E-02		.7645001
.11	.7467747	
.12	.7298562	
.13	.7136817	
.14	.6981883	
.15	.6833128	
.16	.6689923	
.17	.6551638	
.18	.6417643	
.19	.6287308	
.2	.6160003	0.62951853
.21	.6035098	
.22	.5911963	
.23	.5789968	
.24	.5668483	
.25	.5546878	
.26	.5424523	.57965551
.27	.5300788	
.28	.5175044	
.29	.5046658	
.3	.4915003	
.31	.4779448	

* * * * * * * * * * * * * *

I N V E R S E   L A P L A C E   T R A N S F O R M

R O S S   M E T H O D

Maximum order of polynomial = 3
Number of additional derivatives to be included 10
Time interval   .01
Number of timd interval points to calculate 30

INPUT POLYNOMIAL COEFFICIENTS IN PAIRS

Numerator S^0, Denominator S^0,   etc.

8	6
3	11
1	6
0	1

$$\frac{S^2+3S+8}{S^2+6S^2+11S+6} \Rightarrow 3e^{-t} - 6e^{-2t} + 4e^{-3t}$$

			Exact
.01	.9707396		.9707396
.02	.9429176		
.03	.9164742		
.04	.8913521		
.05	.8674957		
.06	.8448519		
.07	.8233691		
.08	.8029978		
8.999999E-02		.7836904	
9.999999E-02		.7654008	
.11	.7480848		
.12	.7316996		
.13	.7162044		
.14	.7015598		
.15	.6877274		
.16	.6746709		
.17	.662355		
.18	.6507459		
.19	.6398109		
.2	.6295187		.62951853
.21	.6198392		
.22	.6107435		
.23	.6022036		
.24	.5941926		
.25	.5866849		
.26	.5796558		.57965551
.27	.5730816		
.28	.5669393		
.29	.5612071		
.3	.5558639		
.31	.5508895		

* * * * * * * * * * * * * *

I N V E R S E   L A P L A C E   T R A N S F O R M

R O S S   M E T H O D

Maximum order of polynomial = 3
Number of additional derivatives to be included 25
Time interval .01
Number of timd interval points to calculate 35

INPUT POLYNOMIAL COEFFICIENTS IN PAIRS

Numerator S^0, Denominator S^0, etc.

$$\frac{S^2+3S+8}{S^3+6S^2+11S+6} \Rightarrow 3e^{-t}-6e^{-2t}+4e^{-3t}$$

8	6
3	11
1	6
0	1

.01	.9707395956010051
.02	.9429175193433215
.03	.9164741402249451
.04	.891351986005785
.05	.8674956709866158
.06	.8448518260948891
.07	.8233690312057582
.08	.802997749628853
.09	.7836902646934652
.1	.7654006183668591
.11	.7480845518424186
.12	.7316994480362763
.13	.7162042759329451
.14	.7015595367222918
.15	.6877272116719594
.16	.6746707116810519
.17	.6623548284625565
.18	.6507456873035884
.19	.6398107013541002
.2	.6295185273962155
.21	.6198390230478094
.22	.6107432053553842
.23	.6022032107326683
.24	.5941922562027017
.25	.5866846019024728
.26	.5796555148104269
.27	.573081233658386
.28	.5669389349906058
.29	.5612067003338393
.3	.5558634844433913
.31	.5508890845912253
.32	.5462641108632294
.33	.5419699574337658
.34	.5379887747866046
.35	.5343034428523046
.36	.5308975450330198

.53089 7544

Other Techniques

Many other techniques for inversion of Laplace transforms have been used and published.

The state-variable technique is generally the most accurate and well-behaved of the methods mentioned.

In circuit analysis, the forementioned implicit integration techniques are superior and should be used.

The Corrington method will be mentioned before leaving this topic because it leads directly to finite-difference equations ie z-transforms.

Corrington Method [23]

Corrington showed that $X(t)$ could also be expressed as a linear homogeneous difference equation resulting in a much simplified calculation of $X(t)$ for $t \geq 0$.

$$X(t) = \sum_{i=1}^{n} (-1)^{i+1} F_{n,i} \, X(t-iT)$$

where $F_{n,i}$ are real constants

Equally spaced data at intervals of T seconds for $2n-1$ successive values of $X(t)$ may be found by using the preceding Ross method, and these used above to find the $F_{n,i}$'s.

For a 3rd order system

$$\begin{bmatrix} X(3T) & X(2T) & X(T) \\ X(4T) & X(3T) & X(2T) \\ X(5T) & X(4T) & X(3T) \end{bmatrix} \begin{bmatrix} F_{31} \\ -F_{32} \\ F_{33} \end{bmatrix} = \begin{bmatrix} X(4T) \\ X(5T) \\ X(6T) \end{bmatrix}$$

$X(t)$ may also be represented in the z-transform domain by a rational algebraic function:

$$X(z) = \frac{N(z)}{D(z)}$$

```
ORDER OF THE DENOMINATOR =  3

INPUT COEFF.          START WITH S^0 COEFFICIENT

COEFF( 0) =       6.000000
COEFF( 1) =      11.000000
COEFF( 2) =       6.000000
COEFF( 3) =       1.000000
INPUT NUMERATOR COEFFICIENTS

COEFF( 0) =        8.000000
COEFF( 1) =        3.000000
COEFF( 2) =        1.000000
COEFF( 3) =        0.000000
COEFF     1      2.940694040386006
COEFF     2     -2.882464396592695
COEFF     3      .9417645333451219
DEN. COEFF. OF Z-TRANSFORM, STARTING WITH S^0

-.9417645333451219
 2.882464396592695
-2.940694040386006
 1

NUM. COEF. OF Z-TRANSFORM, STARTING WITH S^0

-1.459510012846721D-10
 .9707337723854195
-1.969954444785001
 1


 0              1
 1              .9707395956010051
 2              .9429175193433215
 3              .916474140224945
 4              .8913519861475695
 5              .8674956715413853
 6              .8448518274516709
 7              .8233690338605542
 8              .802997754174485
 9              .7836902718100787
 10             .7654006288130501
 11             .7480845664469491
 12             .731699467690345
 13             .7162043015829826
 14             .701559569363251
 15             .6877272523410722
 16             .6746707614520264
 17             .6623548884401885
 18             .6507457586187637
 19             .6398107851591646
 20             .6295186248606907
 21             .6198391353544346
 22             .6107433336964634
 23             .6022033563067046
 24             .5941924202113025
 25             .5866847855475075
```

$N(z)$ and $D(z)$ are both polynomials in z

The coefficients of the $D(z)$ polynomial are equal to the $F_{n,i}$'s with a change of signs.

⊛ Hence, we have a finite difference equation which represents the behavior of the continuous response at discrete instants of time given by $t = nT$.

Nonlinear Behavior

Generally, we try to avoid nonlinear behavior of a phase-locked loop because it seriously impacts the transient performance.

Nonlinear behavior encompasses:

- non-linear loop elements e.g. Sinusoidal phase detector
- Cycle slipping
- Frequency Acquisition, Pull-IN.

Analysis of these phenomena is not easy and in general, necessary inclusion of noise converts the problem into a question of random walks and diffusion processes.

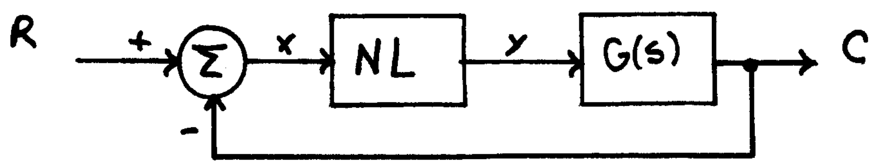
Analysis in these subjects direct attention to

- Wiener Processes
- Chapman-Kolmogorov Equations
- Gaussian-Markov Processes
- Folker-Plank Equations
- etc.

We won't consider noise aspects. For some initial insight into these matters, see [14].

Describing Function (Harmonic Balance) [24], [25]

The describing function method is based upon a Fourier series representation of the nonlinear element in response to a sinusoidal input.



The output of the nonlinearity is generally approximated using only its fundamental frequency component.

Assumptions:

1) All nonlinearities must be in the one NL element
2) The characteristics of N are time-invariant
3) $G(w)$ is lowpass in nature and all harmonic components other than the fundamental are highly attenuated.
4) NL is symmetrical such that no dc component is generated at its output when a sinusoidal input is applied.

Conditions for ascertaining system stability are considerably more complex with non linear systems.

See for example 2nd method Lyapunov
Popov Stability Criteria

Phase-Plane Analysis [14], [24]

Primarily limited to second-order systems with constant inputs.

Returning to the differential equation description of the Type II Loop, we had

$$\frac{d^2\phi}{dt^2} + 2\zeta\omega_n f'(\phi)\frac{d\phi}{dt} + \omega_n^2 f(\phi) = 0$$

or

$$\ddot{\phi} + 2\zeta\omega_n f'(\phi)\,\dot{\phi} + \omega_n^2 f(\phi) = 0$$

Dividing through by $\dot{\phi}$

$$\frac{\ddot{\phi}}{\dot{\phi}} + 2\zeta\omega_n f'(\phi) + \omega_n^2 \frac{f(\phi)}{\dot{\phi}} = 0$$

Note that $\ddot{\phi} = \frac{d}{dt}(\dot{\phi}) = \frac{d}{d\phi}(\dot{\phi})\,\dot{\phi}$

$$\ddot{\phi} = \frac{d\dot{\phi}}{d\phi}\,\dot{\phi}$$

Substituting

$$\frac{d\dot{\phi}}{d\phi} + 2\zeta\omega_n f'(\phi) + \frac{\omega_n^2 f(\phi)}{\dot{\phi}} = 0$$

Note that all explicit time dependence has vanished

$$\frac{d\dot{\phi}}{d\phi} = -2\zeta\omega_n f'(\phi) - \frac{\omega_n^2 f(\phi)}{\dot{\phi}}$$

In the case of a sinusoidal phase detector, $f(\phi) = \sin(\phi)$ and

$$\frac{d\dot{\phi}}{d\phi} = -2\zeta\omega_n \cos(\phi) - \omega_n^2 \frac{\sin(\phi)}{\dot{\phi}}$$

For each value of ϕ (x-axis) and $\dot{\phi}$ (y-axis) we can calculate the slope of the trajectory and thus obtain an overall trajectory path.

Much more detail is included in [14] and in Viterbi's original work.

<u>Note</u>: If the input frequency is being swept, it is advantageous to acquire from a specific side in frequency as the phase plane plot is not symmetric any longer.

Further, the phase-plane analysis can be used with an arbitrary phase detector characteristic $f(\phi)$.

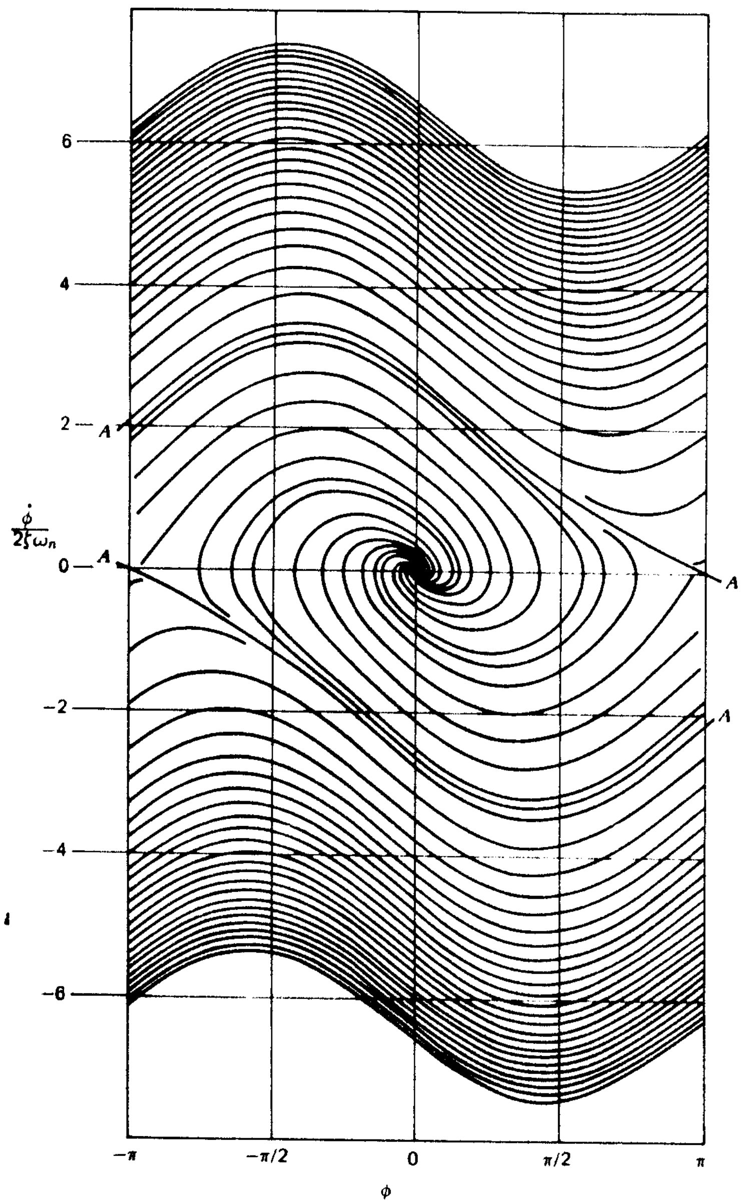


FIGURE 10.8. Phase-plane trajectories for a loop with $F(s)=(1+\tau_2 s)/(\tau_1 s)$ when $\zeta=0.5$ (from A. J. Viterbi; furnished through the courtesy of the Jet Propulsion Laboratory, California Institute of Technology).

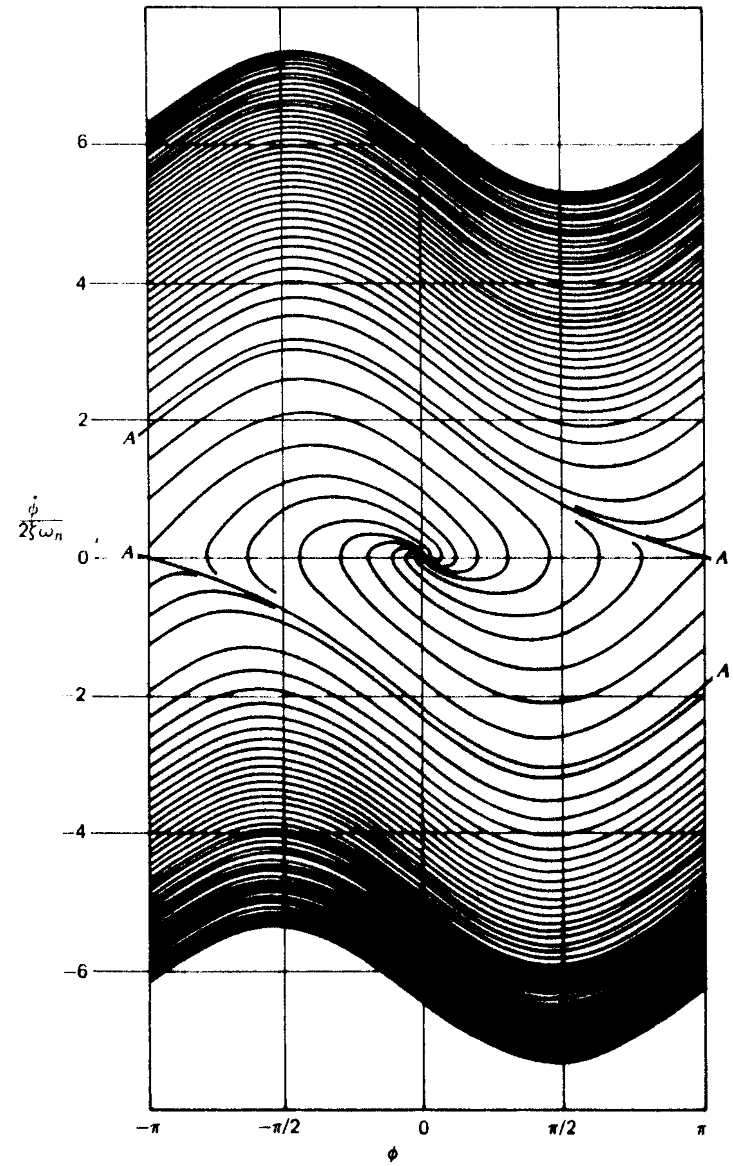


FIGURE 10.9. Phase-plane trajectories for a loop with $F(s)=(1+\tau_2 s)/(\tau_1 s)$ when $\zeta=\sqrt{2}/2$ (from A. J. Viterbi; furnished through the courtesy of the Jet Propulsion Laboratory, California Institute of Technology).

From Viterbi

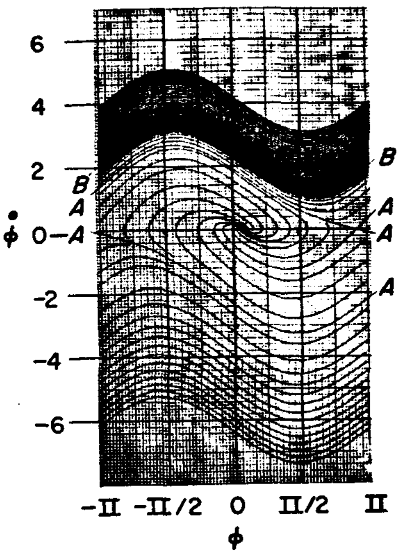


Figure 3.8 **Phase-plane trajectories for second-order loop with linearly varying input frequency ($a' = \frac{1}{2}$, $R'/a' = \frac{1}{4}$).**

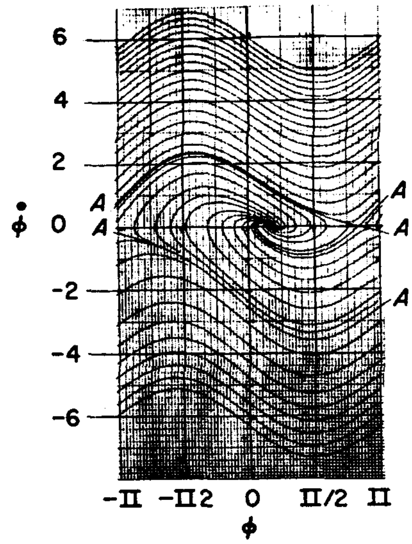


Figure 3.9 **Phase-plane trajectories for second-order loop with linearly varying input frequency ($a' = \frac{1}{2}$, $R'/a' = \frac{1}{2}$).**

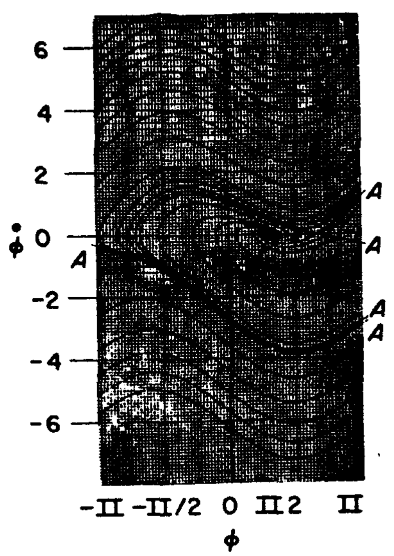


Figure 3.10 **Phase-plane trajectories for second-order loop with linearly varying input frequency ($a' = \frac{1}{2}$, $R'/a' = \frac{3}{2}$).**

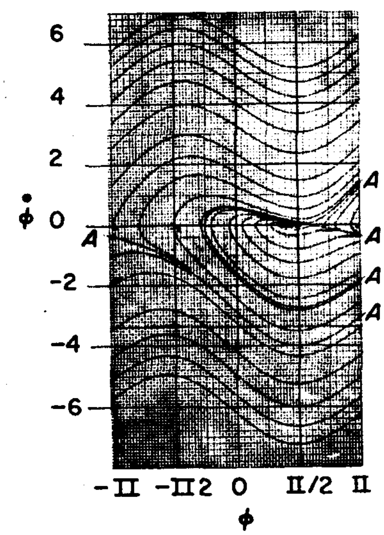


Figure 3.11 **Phase-plane trajectories for second-order loop with linearly varying input frequency ($a' = \frac{1}{2}$, $R'/a' = 0.95$).**

Why Sampled Phase-Locked Loops?

- Any PLL which includes a divide-by-N in the feedback path is fundamentally a sampled system.

- Exclusion of sampled aspects in large bandwidth PLL's results in considerable error.

- Sampled PLL's are capable of faster switching speed per unit loop-bandwidth than any continuous PLL of the same type.

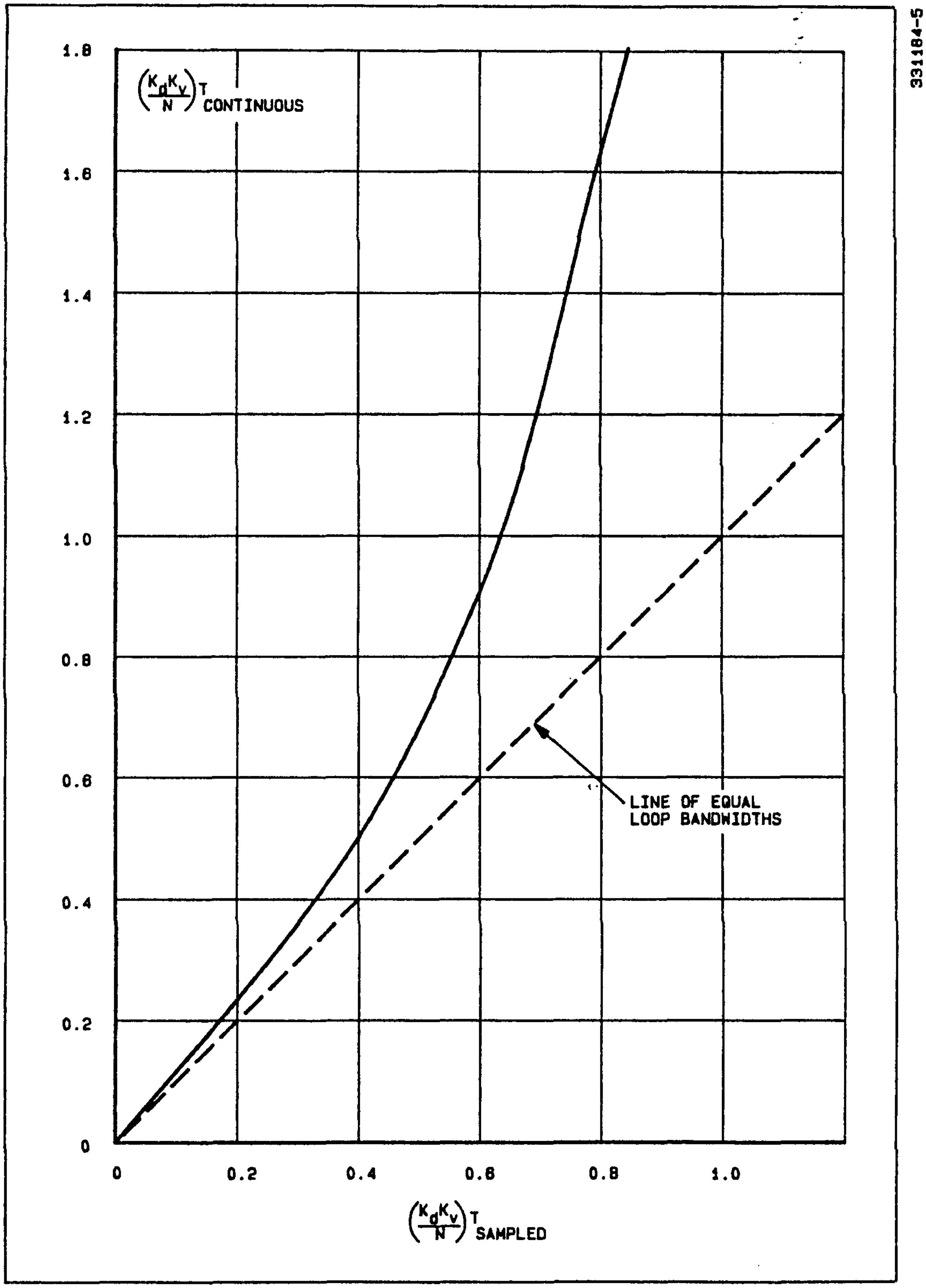


Figure 5. Continuous and Sampled Control System: Phase-Locking Speed [26]

Sampled PLL Model

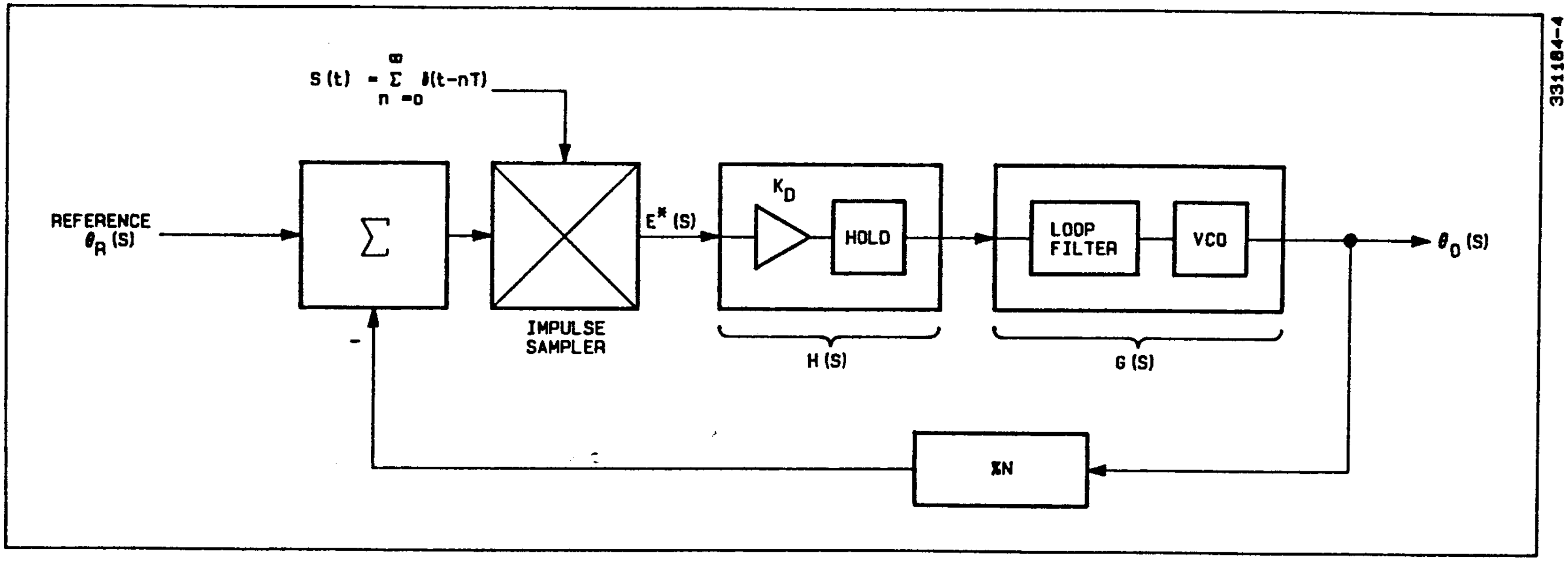
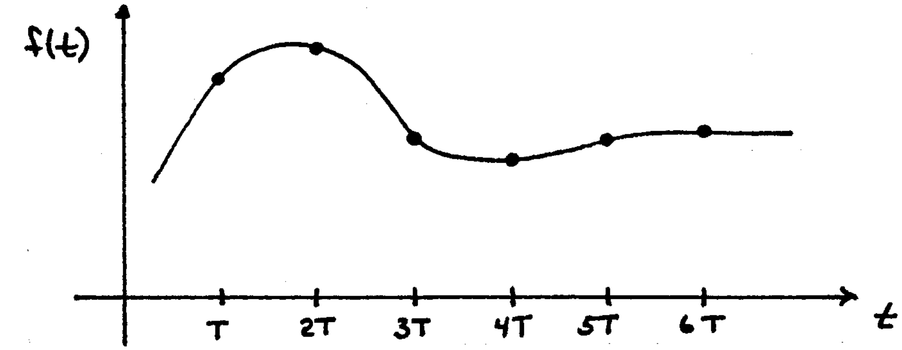
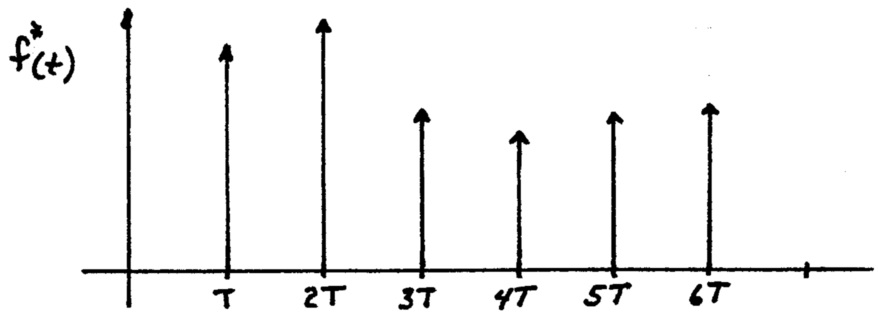


Figure 4. The general Sampled phase-locked loop employs an ideal impulse Sampler which must be followed by some form of ``hold'' device, H(S) [26]

Prelude to Sampled Control Systems





$$\mathcal{L}\left\{f^*(t)\right\} = \sum_{n=0}^{\infty} f(nT)\, e^{-sTn}$$

$$\equiv F(z)$$

Dirac Delta Functions

1) Unit Area $\displaystyle\int_{-\infty}^{\infty} \delta(t)\, dt = 1$

2) Sampling $\displaystyle\int_{-\infty}^{\infty} f(t)\, \delta(t-a)\, dt = f(a)$

Periodic Impulse Sampler

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

$$= \frac{1}{T} \sum_{n=-\infty}^{\infty} e^{jn\omega_s t}$$

$$f^*(t) = f(t)\; p(t)$$

$$= f(t)\, \frac{1}{T} \sum_{n=-\infty}^{\infty} e^{jn\omega_s t}$$

$$\therefore \quad \mathcal{F}\left\{ f^*(t) \right\} = F^*(\omega) = \frac{1}{T} \sum_{n=-\infty}^{\infty} F(\omega - \omega_s n)$$

Sampled PLL- Analysis : Error Response [26]

$$E^*(s) = \left[\theta_R(s) - \theta_o(s)/N\right]^* \quad \text{where } * \text{ denotes ideal impulse sampling}$$

$$= \left[\theta_R(s) - E^*(s)\,G(s)\,H(s)/N\right]^*$$

From a fundamental theorem pertaining to sampled control systems, the sampling process may be brought inside the brackets as: [27]

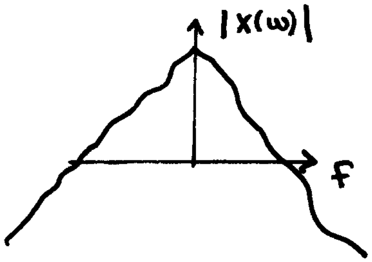
$$E^*(s) = \theta_R^*(s) - E^*(s)\,G^*(s)\,H^*(s)/N$$

$$E^*(s) = \frac{\theta_R^*(s)}{1 + \dfrac{G^*(s)\,H^*(s)}{N}}$$

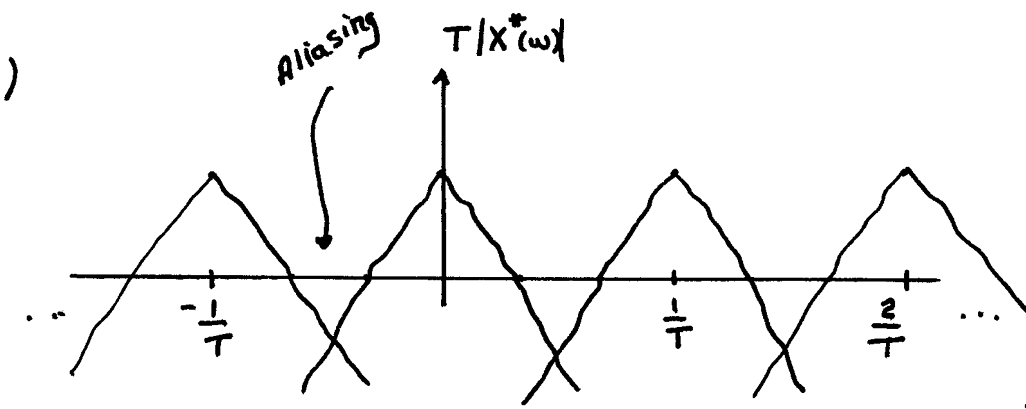
$$= \frac{\dfrac{1}{T}\displaystyle\sum_{n=-\infty}^{\infty} \theta_R(s - jn\,\omega_s)}{1 + \dfrac{1}{NT}\displaystyle\sum_{m=-\infty}^{\infty} G(s+jm\,\omega_s)\,H(s+jm\,\omega_s)}$$

where $\omega_s = \dfrac{2\pi}{T}$

e.g. $X(\omega) \xrightarrow[\text{sampler}]{\text{ideal}} X^*(\omega)$

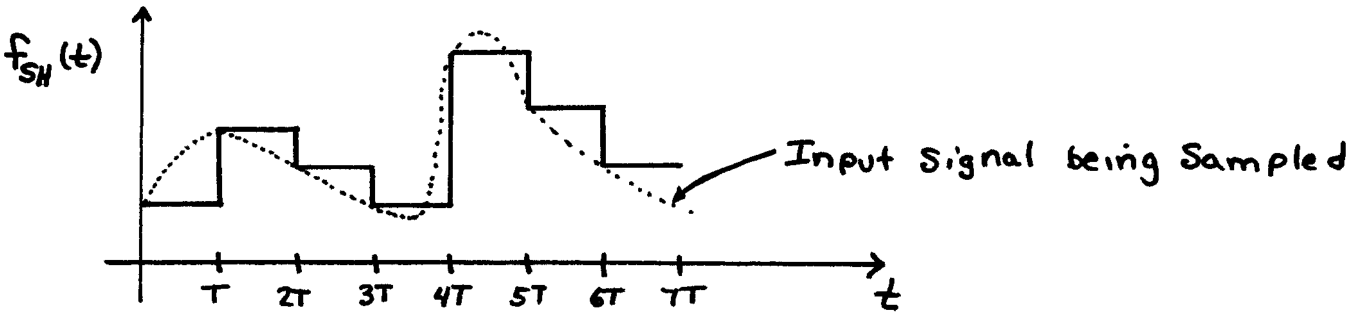


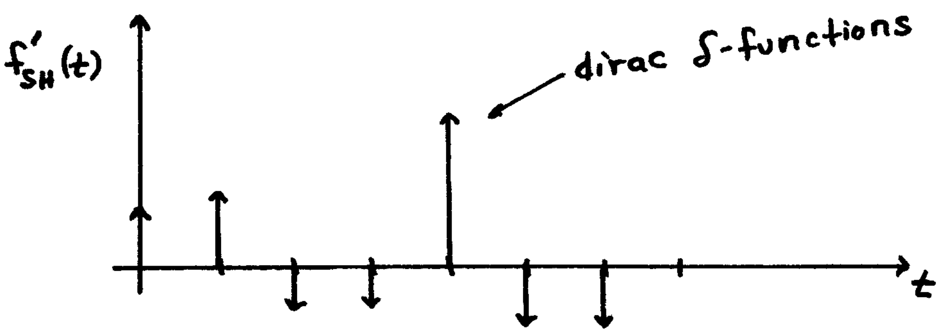
$\Longrightarrow$



Transfer Function: Zero-Order Sample-Hold

Consider the sample-hold output





$$\Rightarrow f'_{SH}(t) = \sum_{n=0}^{\infty} \left[f(t) - f(t-T) \right] \delta(t-nT)$$

$$\mathcal{L}\left\{ f'_{SH}(t) \right\} = \sum_{n=0}^{\infty} \left[F(s) - F(s)\, e^{-sT} \right] \circledast \mathcal{L}\left\{ \delta(t-nT) \right\}$$

$$= F(s)\left(1 - e^{-sT}\right) \circledast \sum_{n=0}^{\infty} e^{-snT}$$

$$= F(s)\left(1 - e^{-sT}\right) \circledast \frac{1}{1 - e^{-sT}} = \int \frac{F(u)\left[1 - e^{-uT}\right]}{1 - e^{(s-u)T}}\, du$$

Let $e^{uT} = z$

$$du = z^{-1} \frac{dz}{T}$$

Then

$$\mathcal{L}\left\{ f'_{SH}(t) \right\} = \frac{1}{T} \int \frac{F(z)\,(1 - z^{-1})\,dz}{z - e^{+sT}} \qquad \text{From Residue Th.}$$

$$= \frac{1}{T} \sum_{n=-\infty}^{\infty} F(u)\,(1 - e^{-uT}) \Big|_{u = s - j\frac{2\pi n}{T}}$$

Removing the derivative can be done by multiplying by $\frac{1}{u}$

$$F_{SH}(s) = \frac{1}{T} \sum_{n=-\infty}^{\infty} F(u) \frac{(1 - e^{-uT})}{u} \Big|_{u = s - j\frac{2\pi n}{T}}$$

If this function is passed thru an ideal LPF (no aliasing)

$$\widetilde{F_{SH}(s)} = \frac{1}{T} F(s) \frac{1 - e^{-sT}}{s} \qquad \text{ie} \quad F(s) \underbrace{\frac{1}{T} \frac{1 - e^{-sT}}{s}}_{\text{lowpass filter equiv. of S/H}}$$

denotes LPF

Sampled PLL Analysis : Z-Transforms [26, 22, 28]

Type I

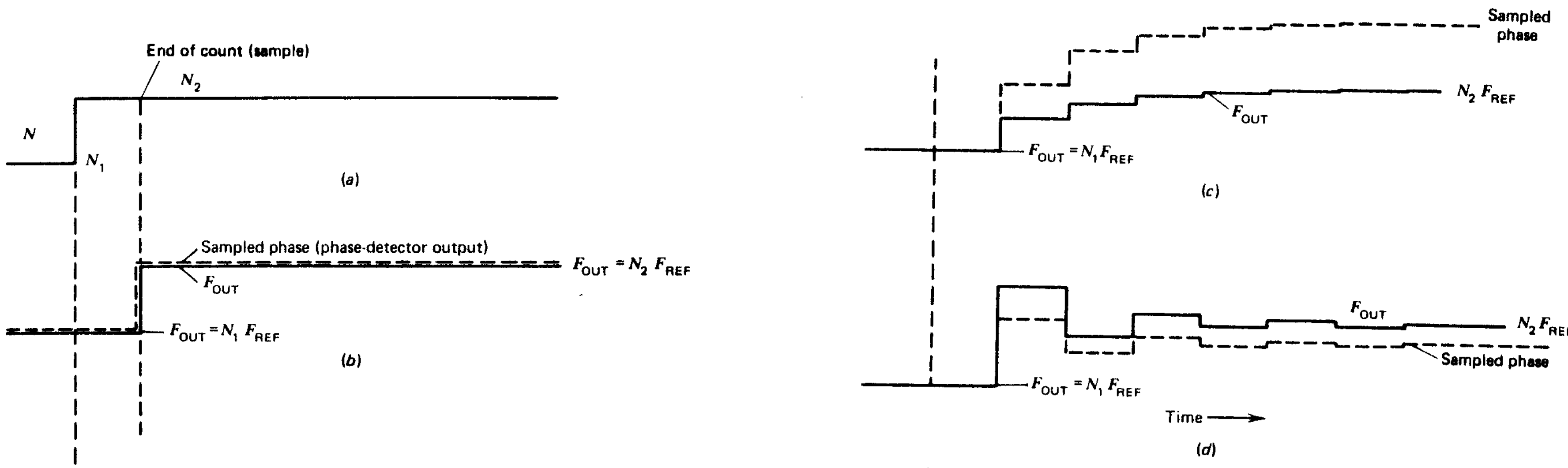
$$G_{OL}(z) = \frac{K}{z-1} \quad \text{where} \quad K = \frac{K_d K_v}{N} T \quad \text{ideal}$$

$$\theta_e(nT) = \frac{2\pi \Delta f}{K_d K_v}\left[1 - (1-K)^n\right] \quad \text{Radians}$$

$$\text{Gain Margin} = -20 \operatorname{Log}\left(\pi \frac{\omega_N}{\omega_S}\right) \; ; \quad \omega_N = \frac{K_d K_v}{N}$$

For $K=1$, Margin = 6 dB

(Analysis excluding aliasing terms would predict a gain margin of 13.1 dB)



Responses of a simple loop: divide ratio at (*a*) and responses with (*b*) optimum gain, (*c*) low gain, and (*d*) high gain.

Sampled PLL Analysis [26]

Type II

$$G_{OL}(z) = \mathcal{Z}\left\{ \frac{1-e^{-sT}}{s} \; \frac{1+s\tau_2}{s\tau_1} \; \frac{K_d}{N} \; \frac{K_v}{s} \right\}$$

$$= (1-z^{-1})\, \mathcal{Z}\left\{ \omega_n^2 \frac{1+s\tau_2}{s^3} \right\} = K \frac{(T/2+\tau_2)z + (T/2-\tau_2)}{(z-1)^2}$$

$$\theta_e(nT) = \frac{2\pi\, \Delta f\, T}{a-b}\left[a^n - b^n \right]$$

$$a, b = -\frac{A}{2} \pm \frac{1}{2}\sqrt{A^2-4B} \qquad \text{where} \quad A = 2 - K\,\frac{T/2+\tau_2}{\tau_1}$$

$$B = 1 + K\,\frac{T/2-\tau_2}{\tau_1}$$

$$K = \frac{K_d K_v}{N}\, T$$

$$\text{Gain Margin} = -20\log\left(2\pi\zeta\frac{\omega_n}{\omega_s}\right) \quad ; \quad \omega_n^2 = \frac{K_d K_v}{N\tau_1}$$

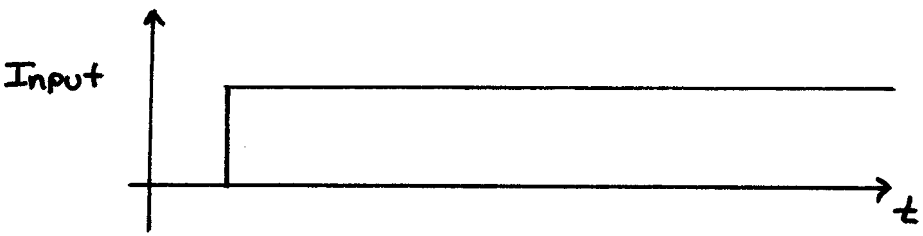
For dead-beat response, $t_{lock} = 2$ samples!

$$\zeta = \frac{1}{2}\omega_n\tau_2 = 0.75 \quad \text{(exactly)}$$

$$\omega_n = \frac{1}{T}$$

Gain Margin ~ 2.5 dB

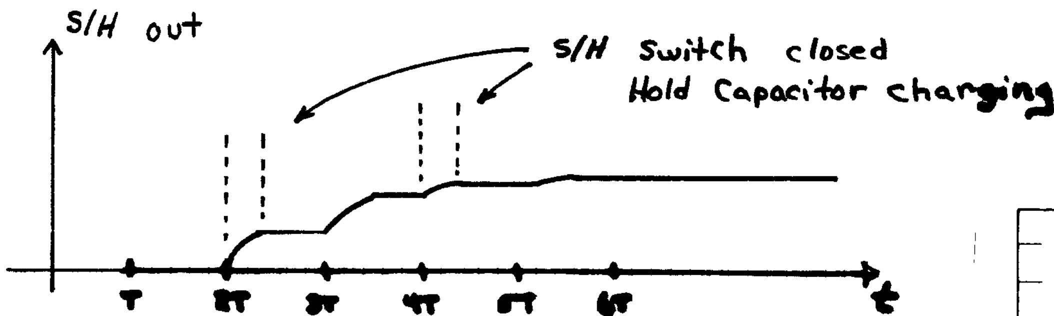
Inefficient Sampling

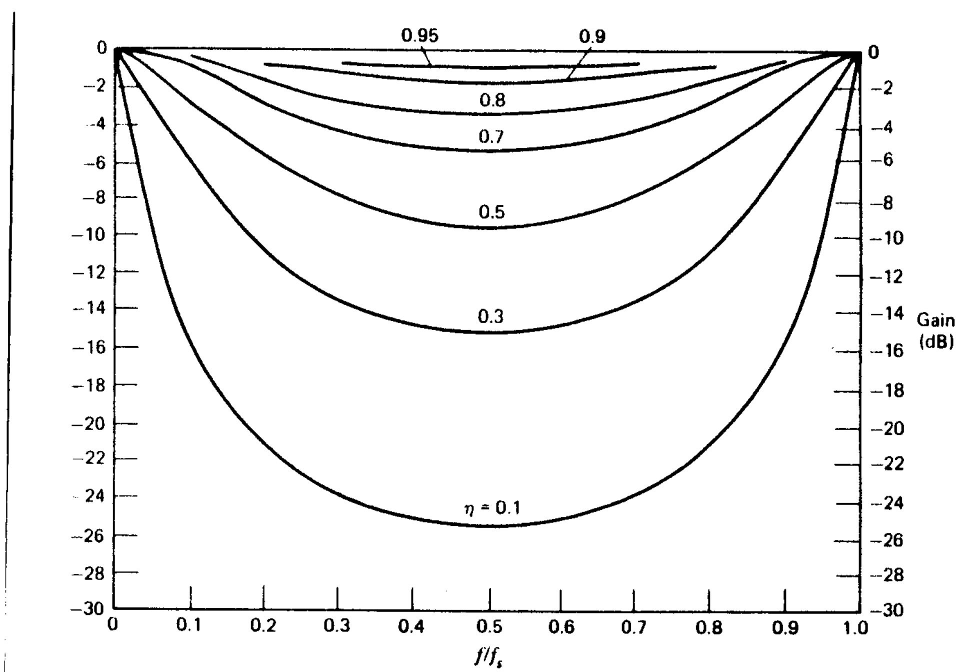


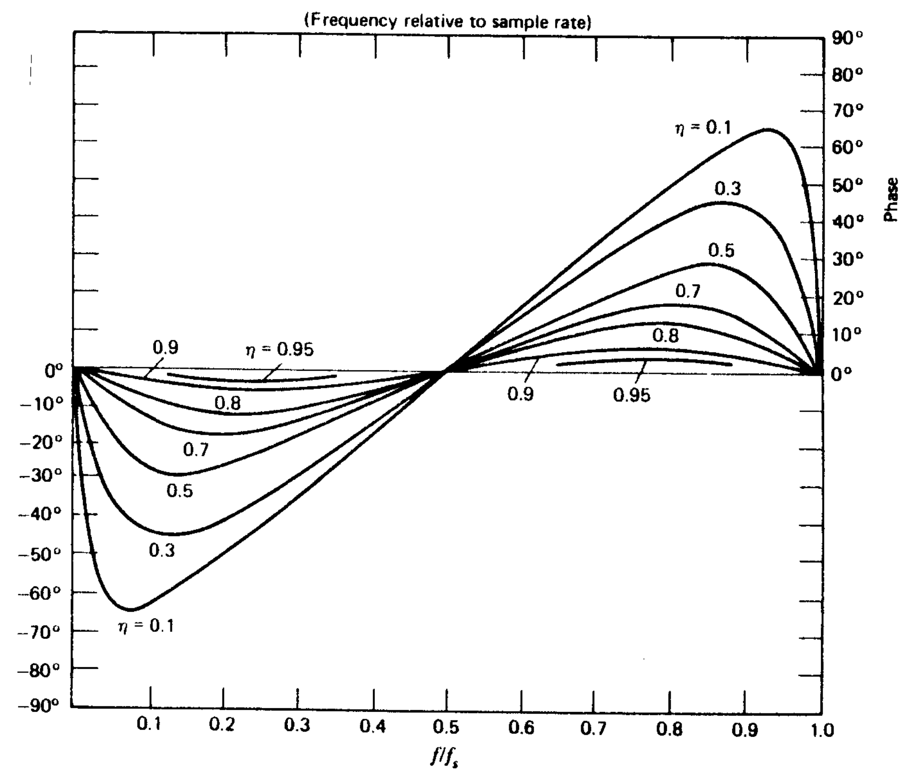
Can show modified S/H transfer function to be

$$\frac{1-e^{-sT}}{s}\,\boxed{\frac{e^{sT}(1-A)}{e^{sT}-A}}$$ efficiency factor

where $A = e^{-\tau_{gate}/\tau_{SH\,RC}}$







Attenuation and phase shift due to efficiency factor.

Type I Sampled Loop Including Time Delay & Inefficient Sampling

The time delay can be automatically accounted for in the analysis by using modified z-transforms.

$$G_{OL}(s) = \frac{K_d}{N} \underbrace{\frac{1-e^{-ST}}{S} \frac{e^{ST}(1-A)}{e^{ST}-A}}_{\text{Inefficient S/H}} \underbrace{\frac{K_V}{S}}_{\text{VCO}} \underbrace{e^{-S\tau_d}}_{\text{Internal Delay}} \underbrace{\frac{1}{1+S\tau_{RC}}}_{\text{Equiv. LPF}} \qquad A = e^{-\tau_{gate}/\tau_{SH\,RC}}$$

$$G_{OL}(z) \cong \frac{K(1-A)}{z-A} \; \frac{z(m-\tau_{RC}/T) + (1+\tau_{RC}/T - m)}{z-1}$$

$$E(z) = \frac{z^3\Delta\phi + z^2(2\pi\Delta F T - \Delta\phi(1+A)) - Az(2\pi\Delta F T - \phi)}{z^3 + z^2(C_{01}-1) + z(C_{02}-C_{01}) - C_{02}}$$

where

$\Delta\phi$ = Initial Step Phase Error at Phase Det., Rad.

ΔF = Step Frequency error at Phase Det, Rad./sec

π = Pi

$C_{01} = K(1-A)(m-\tau_{RC}/T) - 1 - A$

$C_{02} = A + K(1-A)(1+\tau_{RC}/T - m)$

$m = 1 - \tau_d/T$

This analysis was used in a Monte-Carlo simulation to ascertain production yield problems. Bounds for each key parameter had to first be established.

Note: Given $E(z)$, it is a simple matter to calculate $e(nT)$ by long-hand division

e.g. Assume $E(z) = \dfrac{Tz}{(z-1)^2}$ (a linear ramp)

$$E(z) = \frac{Tz}{z^2 - 2z + 1}$$

Divide denominator into numerator repeatedly

$$
\begin{array}{r|l}
 & Tz^{-1} + 2Tz^{-2} + 3Tz^{-3} \\
\hline
z^2 - 2z + 1 & Tz \\
 & \underline{Tz - 2T + Tz^{-1}} \\
 & \quad 2T - Tz^{-1} \\
 & \quad \underline{2T - 4Tz^{-1} + 2Tz^{-2}} \\
 & \qquad 3Tz^{-1} - 2Tz^{-2} \\
 & \qquad \vdots
\end{array}
$$

Since z^{-n} means delay of n T-time units:

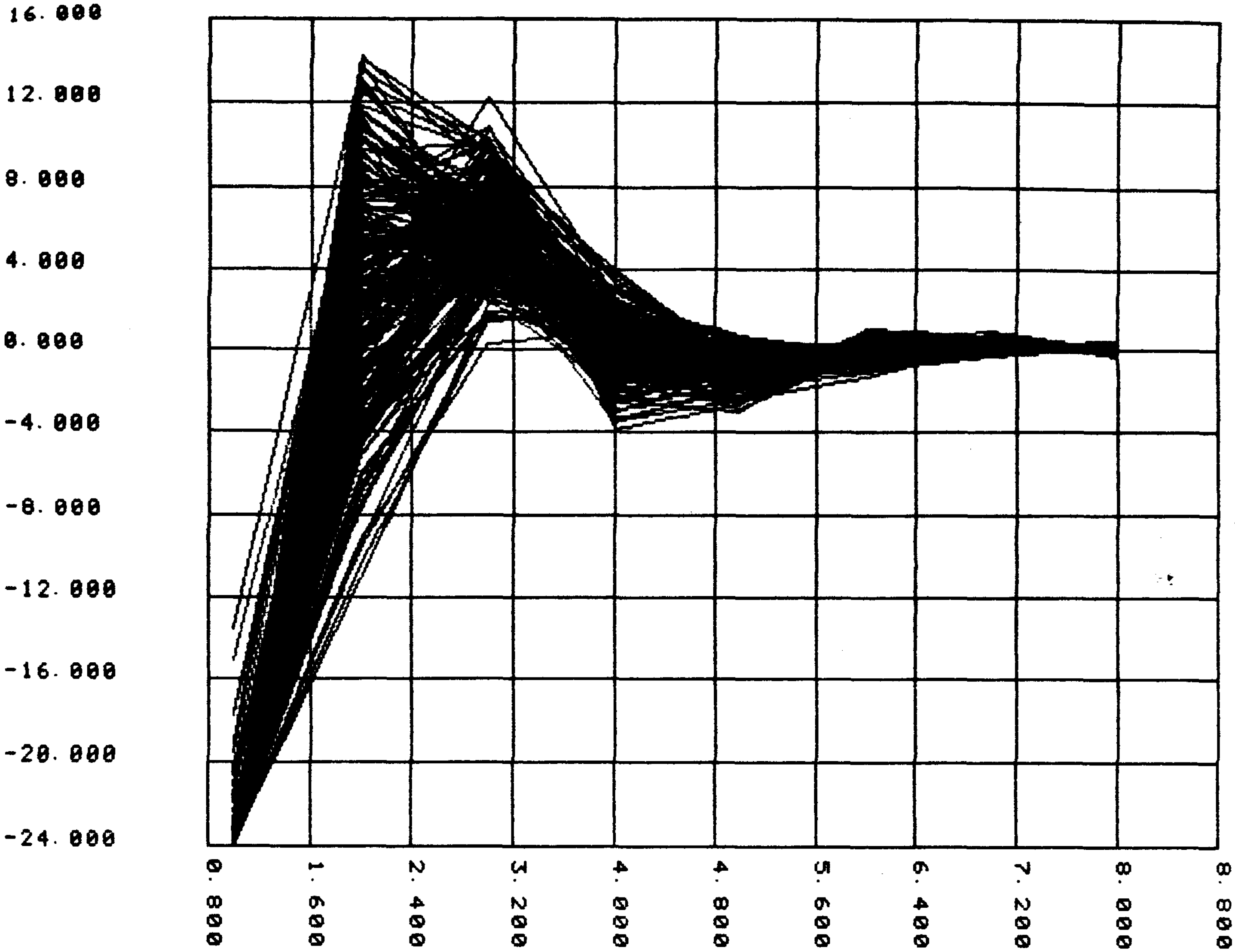
$e(T) = T$

$e(2T) = 2T$

$e(3T) = 3T$

$\vdots$

From above

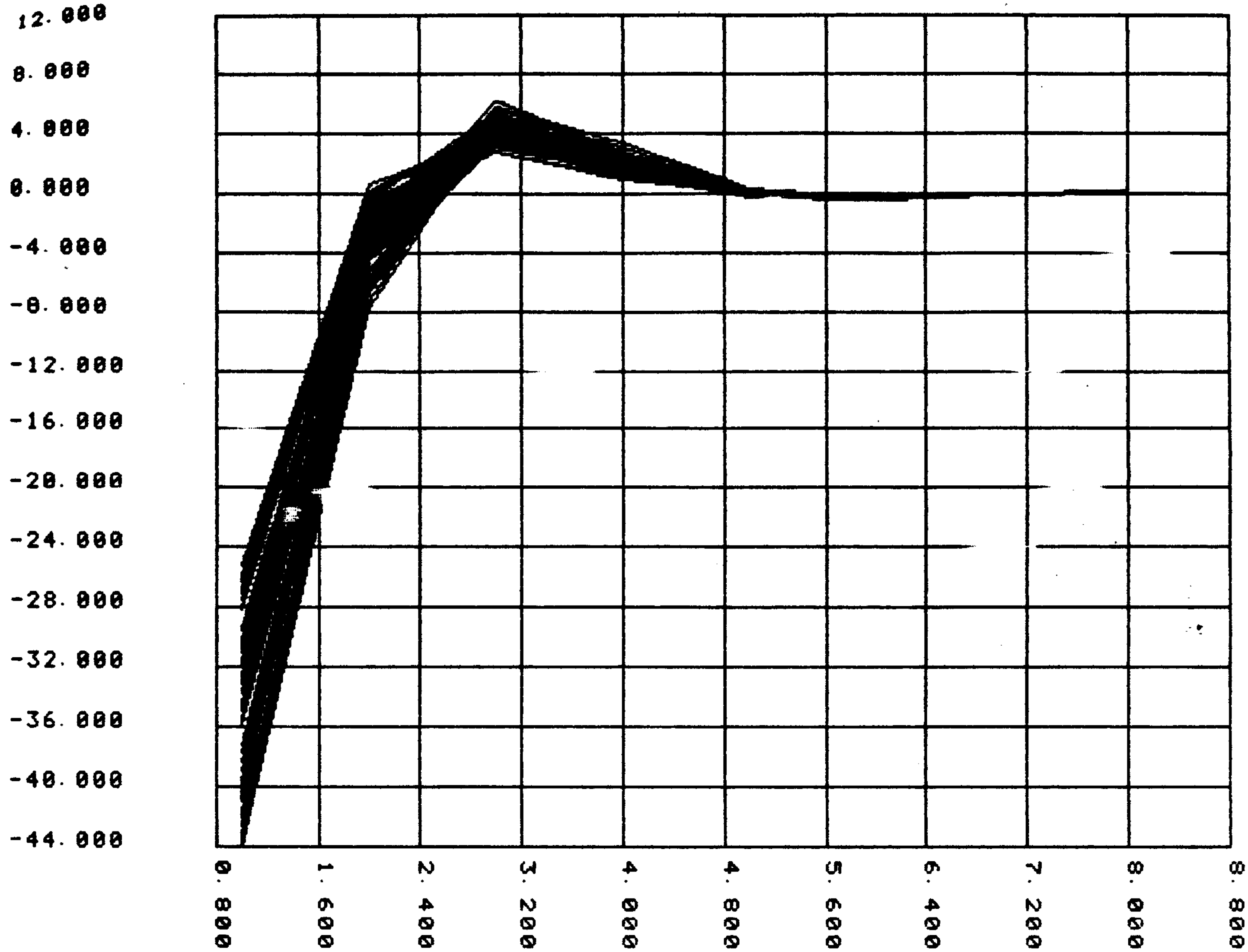


Kd min/max V/Rad. 2.500 3.500
Kv min/max MHz./V 24.000 32.000
Internal delay, nsec min/max 20.000 45.000
Equivalent LPF time constant of loop, min/max nsec 10.000 35.000
Sample/Hold ratio of sample to RC of Bridge min/max 3.000 5.000

Divide by N number = 175
Frequency step MHz, min/max 50.000 50.000
Initial Phase error, deg, min/max 0.000 3599.999

Number of trials was = 200

Figure A1.0



Kd min/max V/Rad. 2.900 3.100
Kv min/max MHz./V 24.000 24.000
Internal delay, nsec min/max 14.000 35.000
Equivalent LPF time constant of loop, min/max nsec 10.000 20.000
Sample/Hold ratio of sample to RC of Bridge min/max 3.000 3.000

Divide by N number = 175
Frequency step MHz, min/max 50.000 50.000
Initial Phase error, deg, min/max 0.000 3599.999

Number of trials was = 75

Figure A1.1

Sampled PLL Anaysis

Type II with Internal Time Delay

$$G_{OL}(z) = \frac{\omega_N^2 \left(a z^2 + bz + c \right)}{(z-1)^3}$$

$$\theta_e(z) = \frac{\Delta\theta z^3 + (2\pi \Delta f T - \Delta\theta) z^2}{z^3 + z^2(a\omega_n^2 - 2) + z(1 + b\omega_n^2) + \omega_n^2 C}$$

$$\omega_n^2 = \frac{K_d K_v}{N \tau_1}$$

$\Delta\theta$: Phase error step, Rad.

Δf = frequency step, Hz

$$m = 1 - \tau_{delay}/T$$

$$a = \frac{(mT)^2}{2} + mT\tau_2$$

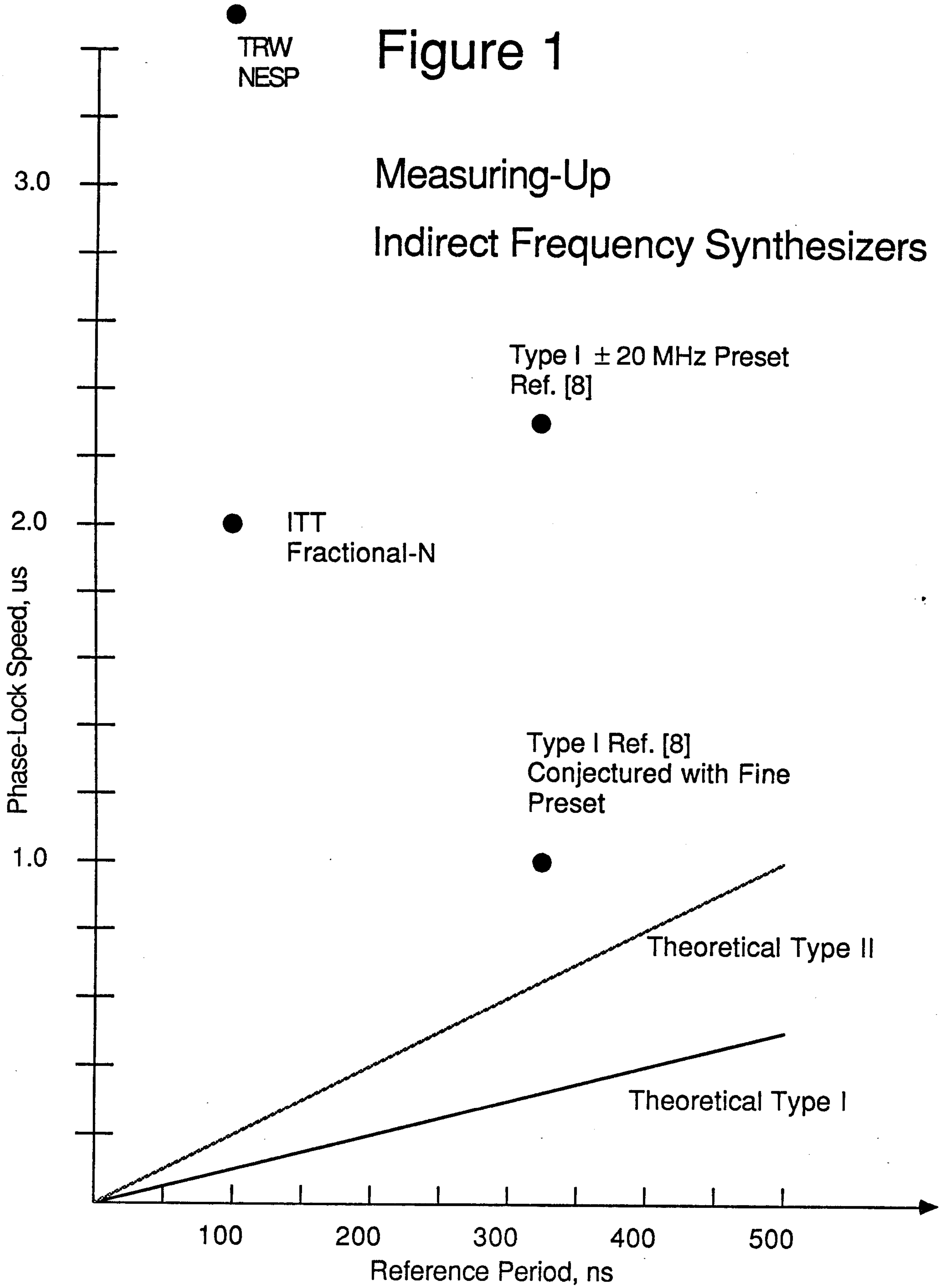
$$b = \frac{T^2}{2}(2m - 2m^2 + 1) - 2m\tau_2 T + \tau_2 T$$

$$c = \frac{T^2}{2}(m-1)^2 + \tau_2 (m-1)T$$

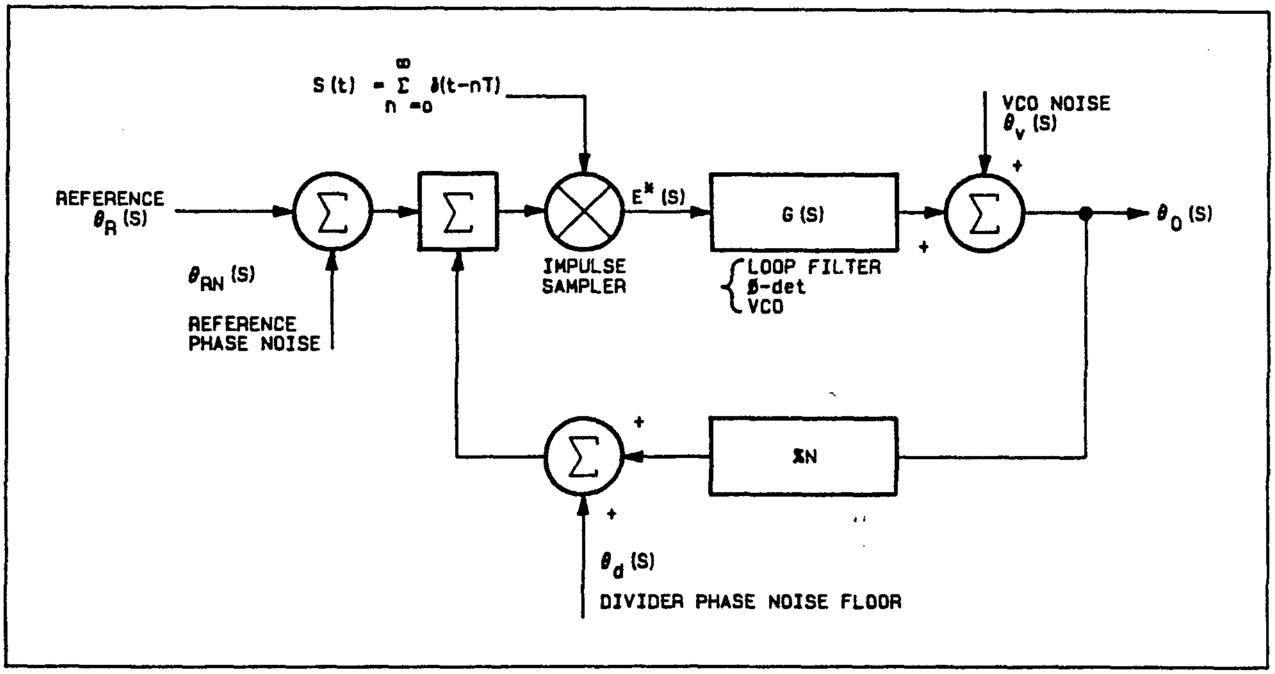
Figure 1

Measuring-Up

Indirect Frequency Synthesizers



Sampled PLLs: Phase Noise Considerations



Phase Noise

Reference noise effect upon the output phase noise:

$$\theta_o(s)\Big|_{Ref} = \frac{\theta^*_{RN}(s) \quad G(s)}{1 + \frac{G^*(s)}{N}}$$

Divider noise effect upon the output phase noise:

$$\theta_o(s)\Big|_{Div} = \frac{\theta^*_d(s) \quad G(s)}{1 + \frac{G^*(s)}{N}}$$

VCO noise effect upon the output phase noise spectrum:

$$\theta_o(s)\Big|_{VCO} = \frac{\theta_v(s) + \left[\theta_v(s)\,G^*(s) - \theta^*_v(s)\,G(s)\right]/N}{1 + \frac{G^*(s)}{N}}$$

Not necessarily 0

$$\approx \frac{\theta_v(s)}{1 + \frac{G^*(s)}{N}}$$

Simplified Phase Noise : (Type I Loop Only, Ideal)

Reference Noise

$$\frac{S_o(\omega)}{S_{RN}(\omega)} = \frac{16\,K^2 N^2 \sin^4(\omega T/2)}{(\omega T)^4 \left(K^2+2-2K+2(K-1)\cos(\omega T)\right)} \quad ;\ K=\frac{K_d K_v T}{N}$$

$$= N^2 \left[\frac{\sin(\omega T/2)}{(\omega T/2)}\right]^4 \quad \text{for } K \equiv 1$$

Divider Noise

Same as reference case

VCO Noise

$$\frac{S_o(\omega)}{S_v(\omega)} = \frac{4\sin^2(\omega T/2)}{K^2+2-2K+2(K-1)\cos(\omega T)}$$

$$= 4\sin^2\left(\frac{\omega T}{2}\right) \quad \text{for } K \equiv 1$$

Conclusion :

* This discussion has only scratched the surface of the phase-locked loop concept.

* I only hope that you have seen some new material which will cause you to ask "what if" in your next design.

See Appendix D for a discussion of the Ø/f detector

See Appendix C for a discussion of a possible harmonic sampler.

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