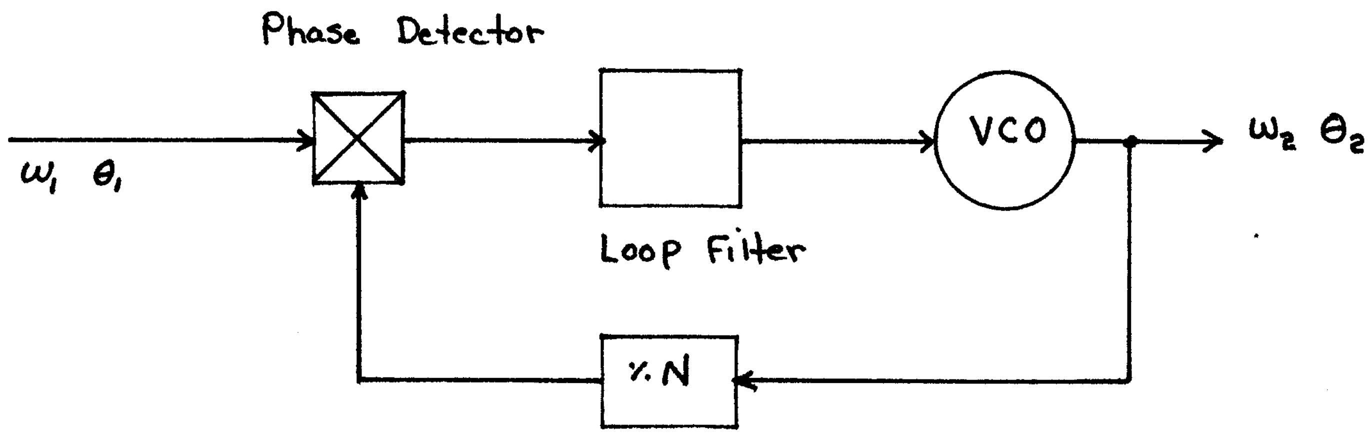
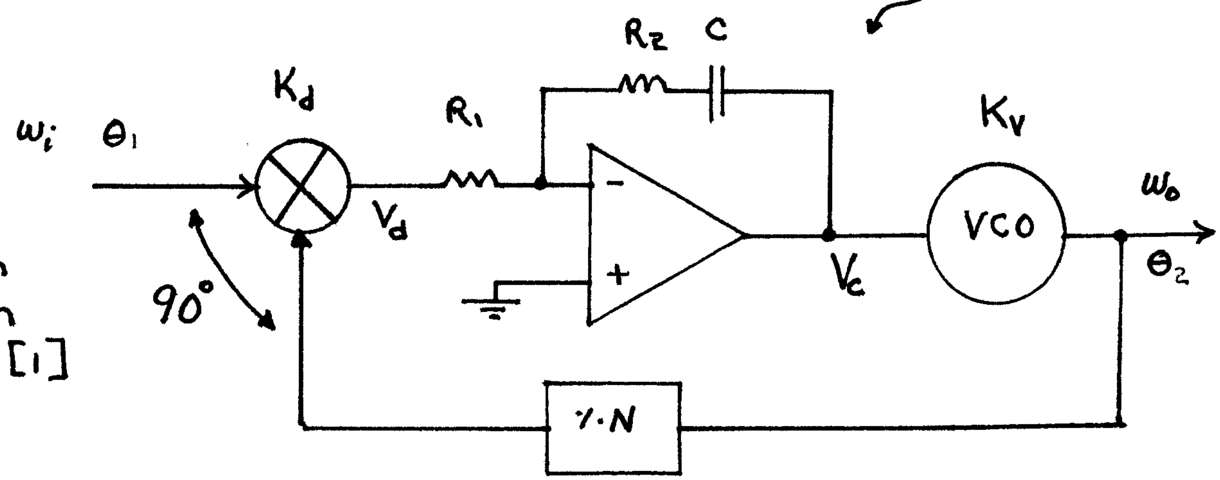
The Phase-Locked Loop Concept for Frequency Synthesis



Consider a Type II PLL:

Classical Lead-Lag network



Loop Type : Number of ideal integrators in the open-loop gain transfer function [1]

θ_1, θ_2 initial phases

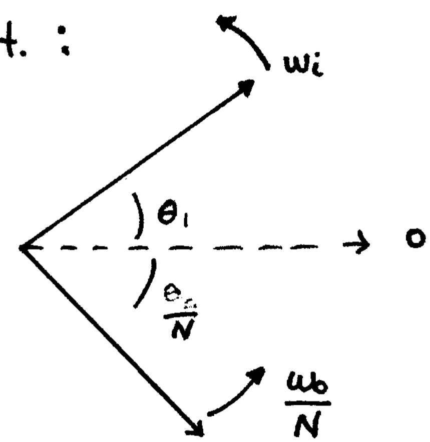
Standard Servo-System Nomenclature:

Phasor Concept

Natural Frequency $\quad \omega_n = \sqrt{\dfrac{K_d K_v}{N \tau_1}}$

At ϕ-det. :

$t \equiv 0$



Damping Factor $\quad \zeta = \frac{1}{2} \omega_n \tau_2$

Time Constants $\quad \tau_1 = R_1 C$
$\tau_2 = R_2 C$

Time-Domain Analysis (Noise-Free Type II) [2]

The VCO acts as an ideal phase integrator:

Total output phase $\theta_o = \omega_c t + \int_0^t K_v V_c \, dt + \theta_{oo}$

ω_c = VCO center freq.

K_v = VCO sensitivity Rad/sec/Volt

Assume $N = 1$

Mixer - Phase Detector

Multiplying inputs together and neglecting high frequency terms,

$$V_d = K_d \sin(\theta_i - \theta_o)$$

θ_i = total input phase $= \omega_i t + 0$ (offset)

$$= K_d \sin\left[(\omega_i - \omega_c)t - \theta_{oo} - \int_0^t K_v V_c \, dt \right]$$

$$= K_d \sin \phi \qquad \text{where} \quad \phi = \theta_i - \theta_o$$

Lead-Lag Filter

Straight forward to show that

$$\tau_1 \dot{V}_c = -V_d - \tau_2 \dot{V}_d$$

Taking additional derivatives

$$\dot{\phi} = \dot{\theta}_i - \dot{\theta}_o = \omega_i - \omega_c - K_v V_c$$

$$\ddot{\phi} = -K_v \dot{V}_c = -\frac{K_v}{\tau_1}\left[V_d + \tau_2 \dot{V}_d \right]$$

$$\dot{V}_d = K_d \cos\phi \; \dot{\phi}$$

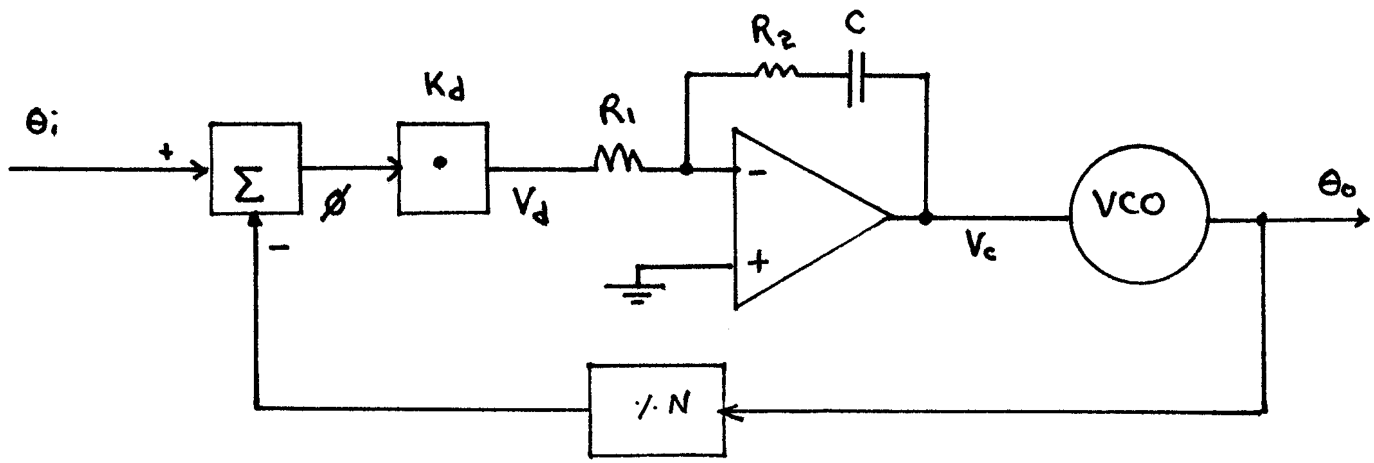
$$\ddot{\phi} = -\frac{K_v}{\tau_1}\left[K_d \sin\phi + \tau_2 K_d \cos\phi \; \dot{\phi} \right]$$

Hence

$$\frac{d^2\phi}{dt^2} + \frac{K_d K_v}{\tau_1}\tau_2 \cos\phi \frac{d\phi}{dt} + \frac{K_d K_v}{\tau_1}\sin\phi \equiv 0 \qquad \left\{ \begin{array}{l} \text{case of initial} \\ \text{phase \& freq. offset} \end{array} \right\}$$

Loop Analysis

Ideal Continuous Type II PLL (with linear approximation)



$$V_d = K_d \phi = K_d \left(\theta_i - \frac{\theta_o}{N} \right) = K_d \left[\theta_i - \frac{K_v}{SN} V_c(s) \right]$$

$$= K_d \left[\theta_i - \frac{K_v}{NS} \frac{1 + S\tau_2}{S\tau_1} V_d \right]$$

$$V_d = \frac{K_d \theta_i}{1 + \frac{K_d K_v}{N\tau_1} \frac{1 + S\tau_2}{S^2}}$$

OR $$\phi(s) = \frac{S^2 \theta_i(s)}{S^2 + \frac{K_d K_v \tau_2}{N\tau_1} S + \frac{K_d K_v}{N\tau_1}} = \frac{S^2 \theta_i(s)}{S^2 + 2\zeta\omega_n S + \omega_n^2}$$

$$\omega_n^2 = \frac{K_d K_v}{N\tau_1} \; ; \; \zeta = \frac{1}{2} \omega_n \tau_2$$

Important Synthesizer Specifications

- Bandwidth
- Channel Spacing
- Phase Noise
- Spurs
- Switching Speed
- Power Consumption
- Post-Tuning Drift
- EMI

Type II PLL Bandwidth Relationships [1]

0-db open-loop BW $$\omega_{0db} = \omega_n \left[2\xi^2 + \sqrt{4\xi^4 + 1} \right]^{1/2}$$

3 db closed-loop BW $$\omega_{3db} = \omega_n \left[1 + 2\xi^2 + \sqrt{(2\xi^2+1)^2 + 1} \right]^{1/2}$$

Phase Margin $$\varphi_M = \tan^{-1}\left[\frac{2\omega_0 \xi}{\omega_n} \right]$$

where $$\omega_0 = \omega_n \left[2\xi^2 + \sqrt{4\xi^4 + 1} \right]^{1/2}$$

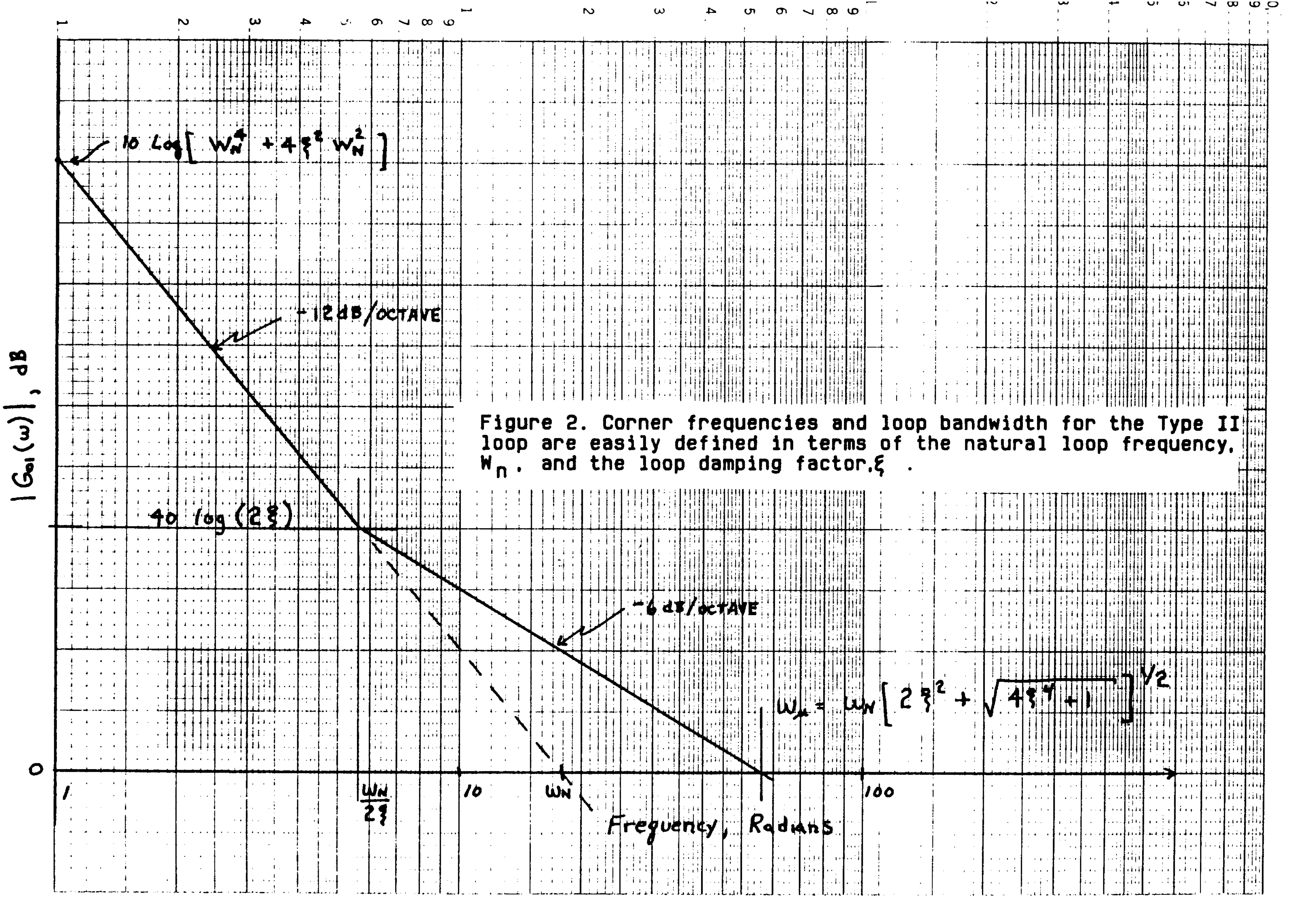


Figure 2. Corner frequencies and loop bandwidth for the Type II loop are easily defined in terms of the natural loop frequency, W_n, and the loop damping factor, ξ.

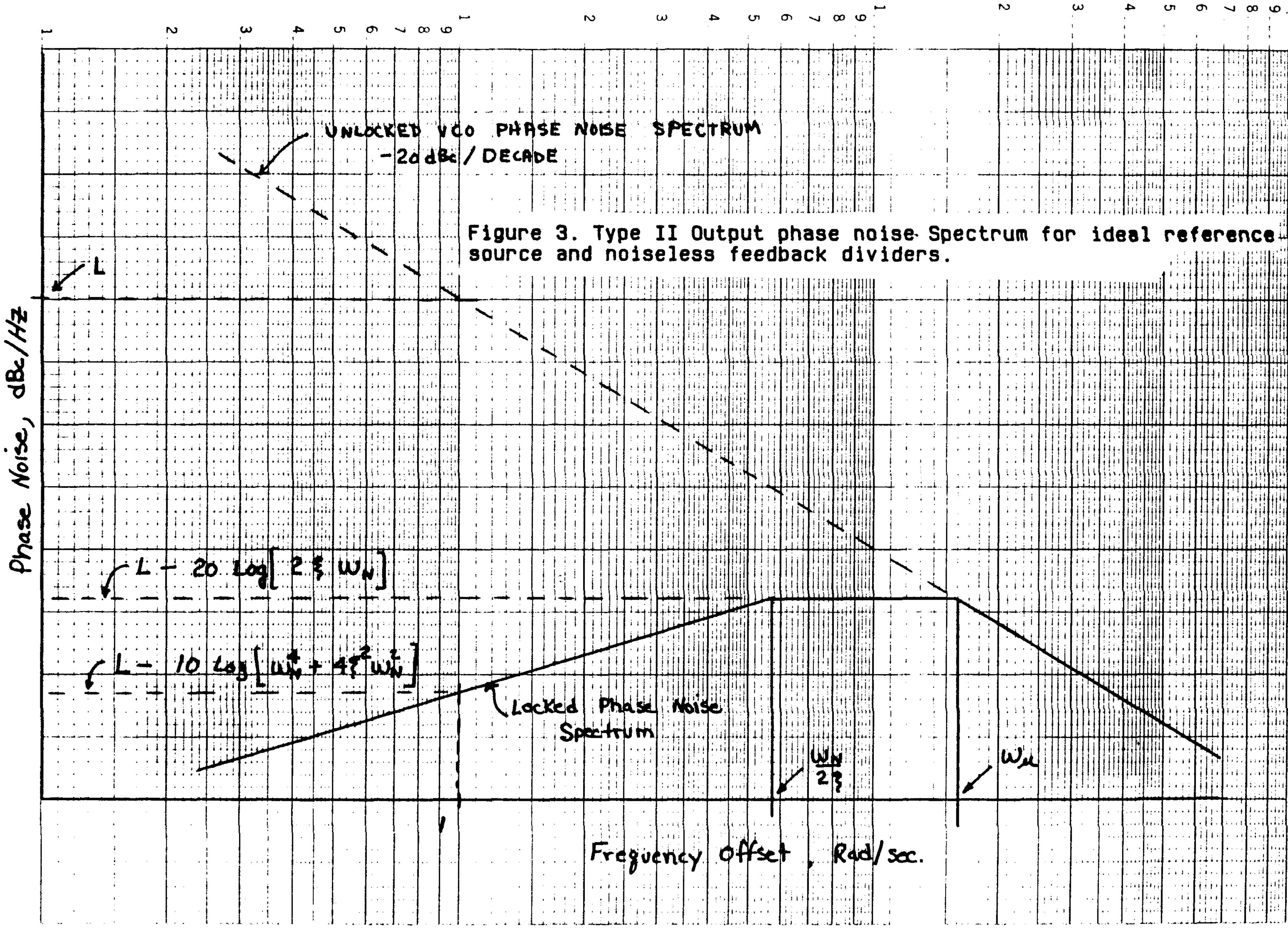
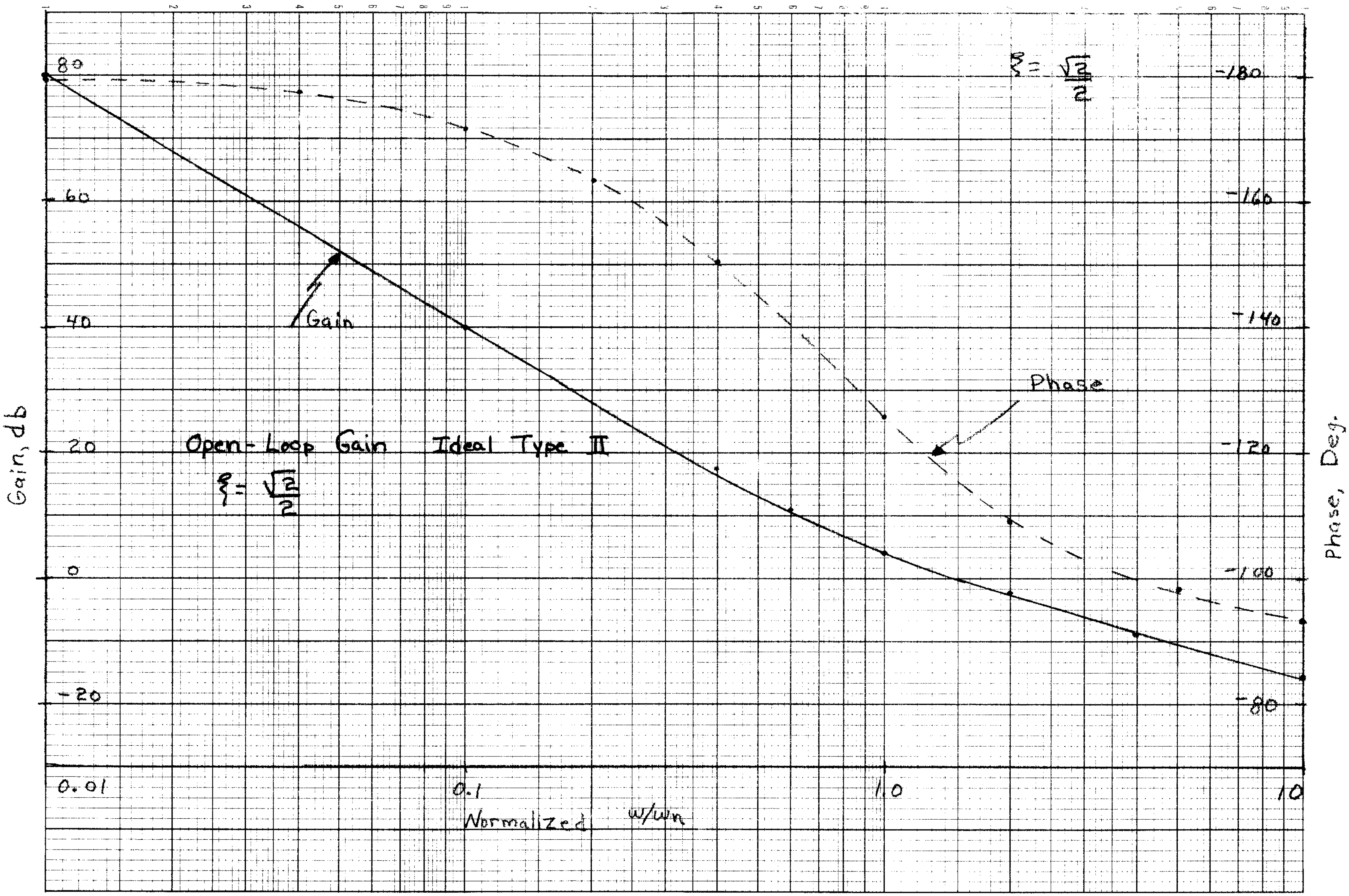


Figure 3. Type II Output phase noise Spectrum for ideal reference source and noiseless feedback dividers.



Open-Loop Gain Ideal Type II
ξ = √2/2
ξ = √2/2
Gain
Phase
Gain, db
Phase, Deg.
80
60
40
20
0
-20
-180
-160
-140
-120
-100
-80
0.01
0.1
1.0
10
Normalized ω/ωn

From:

"Noise Property Analysis
Enhances PLL Designs"

Larry Martin
EDN Sept. 16, 1981

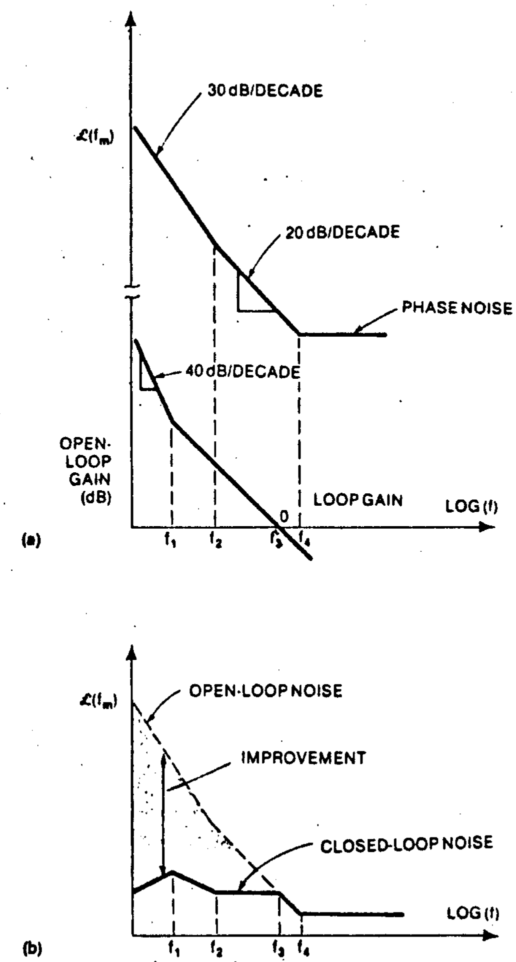


Fig 6—These Bode plots *of open-loop phase noise and gain* **(a)** *and closed-loop VCO noise* **(b)** *show the reduction of oscillator phase noise inside the loop bandwidth by the available loop gain at any offset frequency.*

From:

"Noise Property Analysis Enhances PLL Designs"

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NOISE DATA

OFFSET FREQUENCY (Hz)	$\mathcal{L}(f_m)$ VCO (dBc)	REFERENCE (dBc)	DIVIDER (dBc)
10	22	87	99
100	52	92	115
1000	82	97	124
10,000	112	102	127
100,000	132	102	147

$\overline{e_{VT}} = 30\ nV/\sqrt{Hz}$

$K_p = [0.4/(2\pi)]V/RAD$

$N = 240$

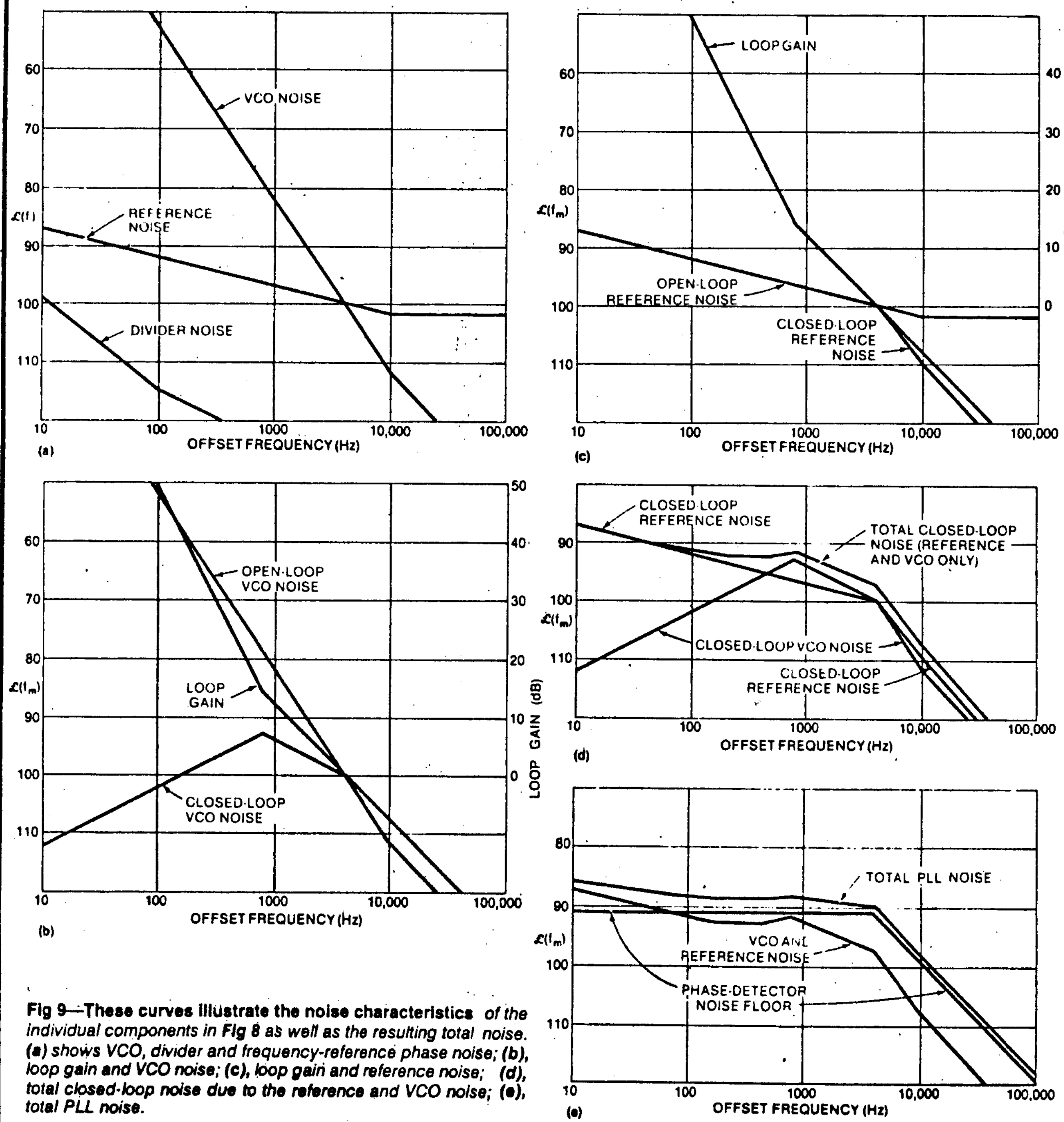


Fig 9—These curves illustrate the noise characteristics *of the individual components in Fig 8 as well as the resulting total noise.* ***(a)*** *shows VCO, divider and frequency-reference phase noise;* ***(b),*** *loop gain and VCO noise;* ***(c),*** *loop gain and reference noise;* ***(d),*** *total closed-loop noise due to the reference and VCO noise;* ***(e),*** *total PLL noise.*

Table 4.1 Transient Phase Error of Second-Order Loop, $\theta_e(t)$ (in rad) (high loop gain; $K_o K_d >> \omega_n$)

	Phase Step ($\Delta\theta$ rad)	Frequency Step ($\Delta\omega$ rad/sec)	Frequency Ramp ($\Delta\dot{\omega}$ rad/sec^2)
$\zeta<1$	$\Delta\theta\left(\cos\sqrt{1-\zeta^2}\,\omega_n t - \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\sqrt{1-\zeta^2}\,\omega_n t\right)e^{-\zeta\omega_n t}$	$\frac{\Delta\omega}{\omega_n}\left(\frac{1}{\sqrt{1-\zeta^2}}\sin\sqrt{1-\zeta^2}\,\omega_n t\right)e^{-\zeta\omega_n t}$	$\frac{\Delta\dot{\omega}}{\omega_n^2} - \frac{\Delta\dot{\omega}}{\omega_n^2}\left(\cos\sqrt{1-\zeta^2}\,\omega_n t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\sqrt{1-\zeta^2}\,\omega_n t\right)e^{-\zeta\omega_n t}$
$\zeta=1$	$\Delta\theta(1-\omega_n t)e^{-\omega_n t}$	$\frac{\Delta\omega}{\omega_n}(\omega_n t)e^{-\omega_n t}$	$\frac{\Delta\dot{\omega}}{\omega_n^2} - \frac{\Delta\dot{\omega}}{\omega_n^2}(1+\omega_n t)e^{-\omega_n t}$
$\zeta>1$	$\Delta\theta\left(\cosh\sqrt{\zeta^2-1}\,\omega_n t - \frac{\zeta}{\sqrt{\zeta^2-1}}\sinh\sqrt{\zeta^2-1}\,\omega_n t\right)e^{-\zeta\omega_n t}$	$\frac{\Delta\omega}{\omega_n}\left(\frac{1}{\sqrt{\zeta^2-1}}\sinh\sqrt{\zeta^2-1}\,\omega_n t\right)e^{-\zeta\omega_n t}$	$\frac{\Delta\dot{\omega}}{\omega_n^2} - \frac{\Delta\dot{\omega}}{\omega_n^2}\left(\cosh\sqrt{\zeta^2-1}\,\omega_n t + \frac{\zeta}{\sqrt{\zeta^2-1}}\sinh\sqrt{\zeta^2-1}\,\omega_n t\right)e^{-\zeta\omega_n t}$
	Steady-state error $=0$	Steady-state error $=\frac{\Delta\omega}{K_v}$ (not included above)	Steady state error $=\frac{\Delta\dot{\omega}t}{K_v}+\frac{\Delta\dot{\omega}}{\omega_n^2}$ ($\Delta\dot{\omega}t/K_v$ not included above)

From Gardner [1]

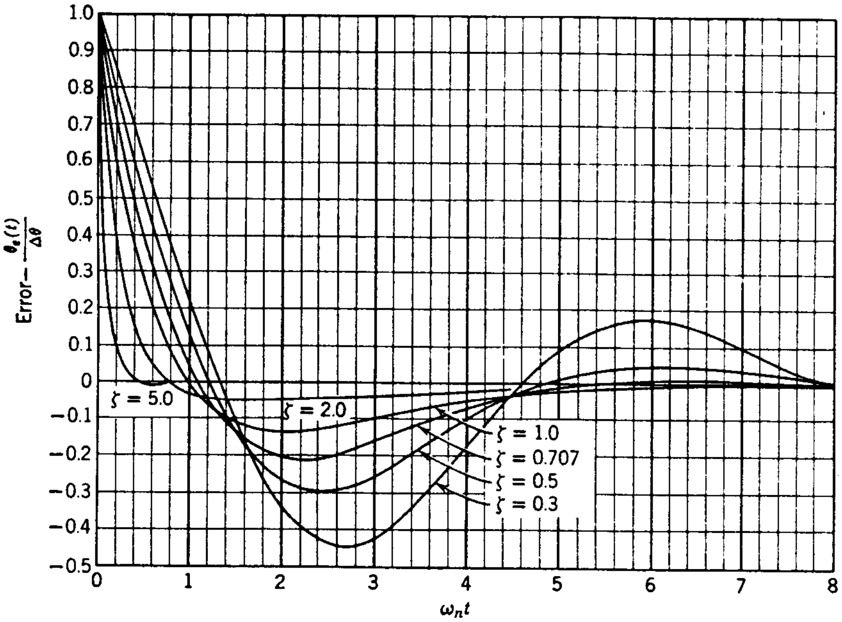


Figure 4.1 Phase error $\theta_e(t)$ due to a step in phase $\Delta\theta$. From Ref. 1 by permission of L. A. Hoffman.

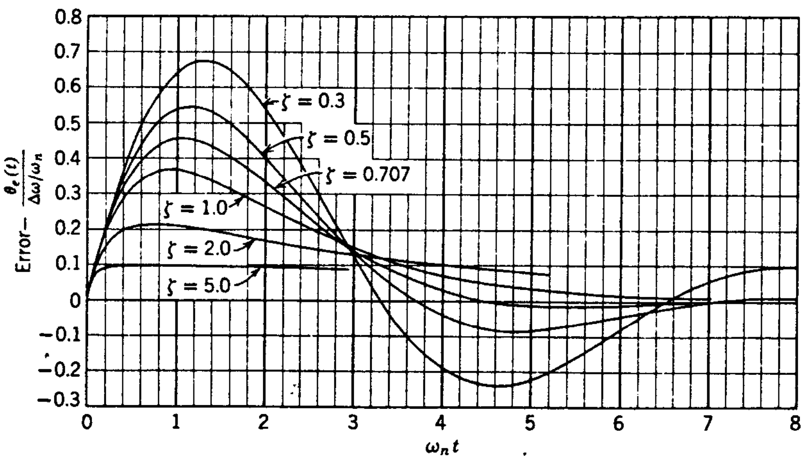
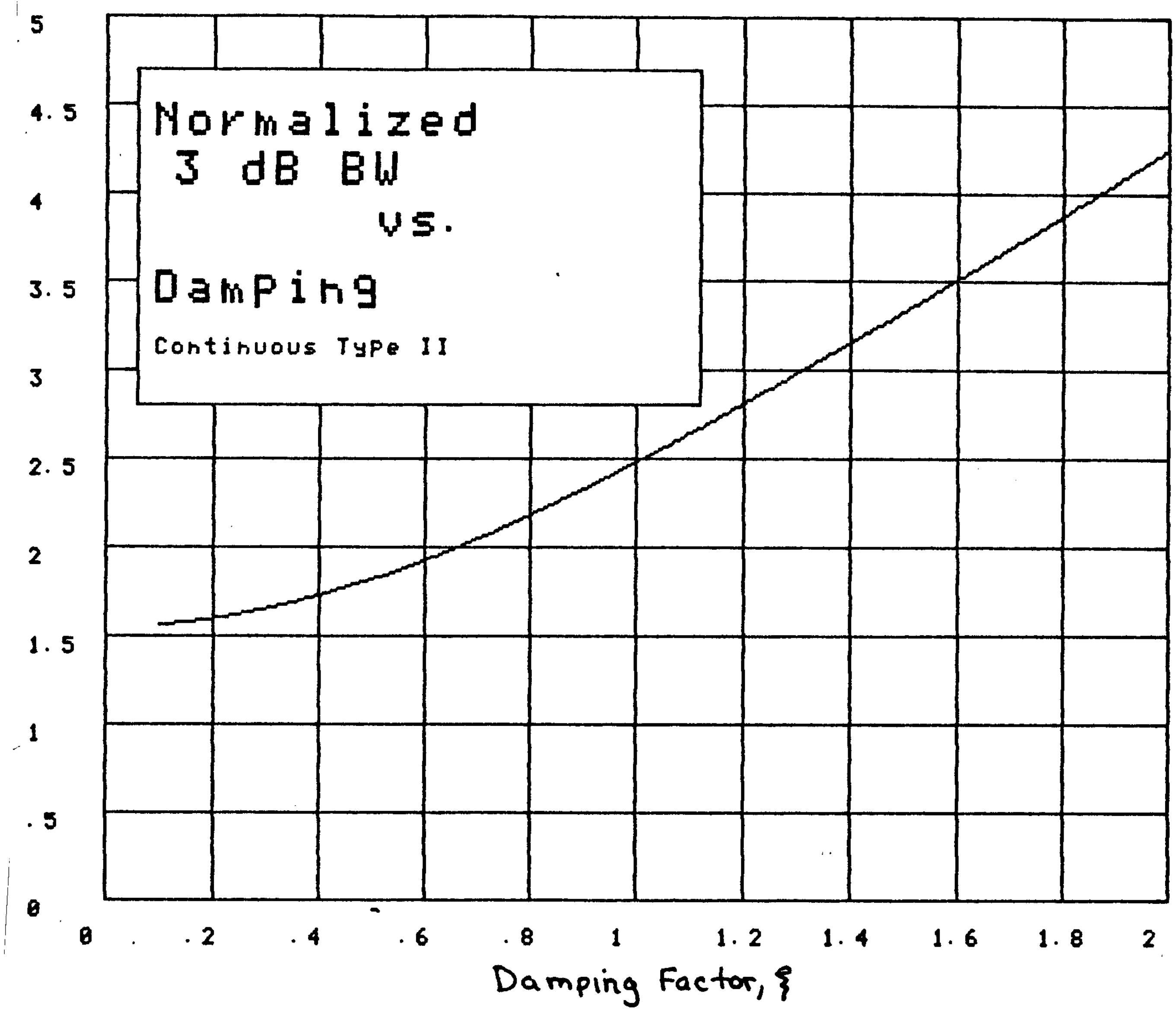
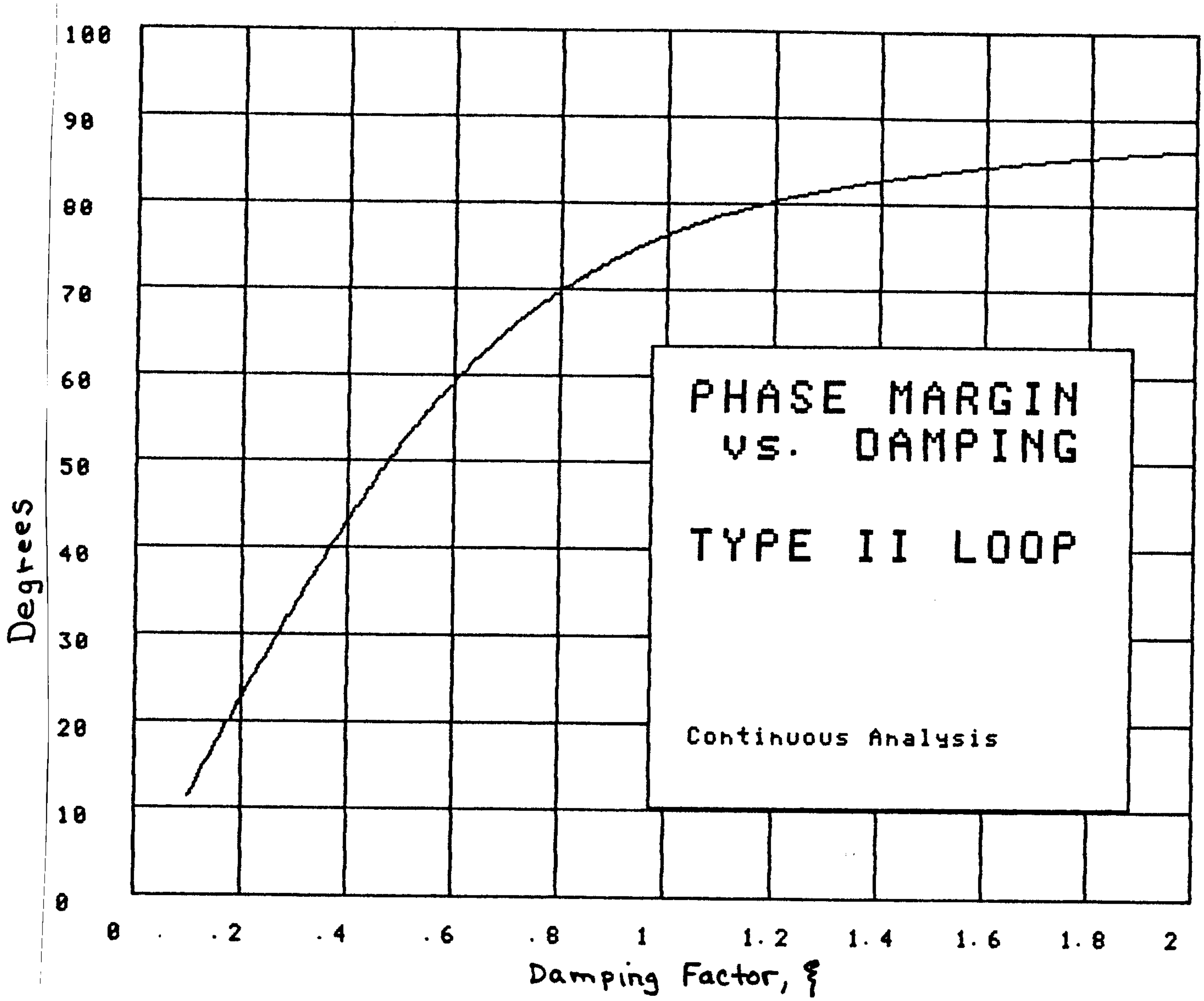
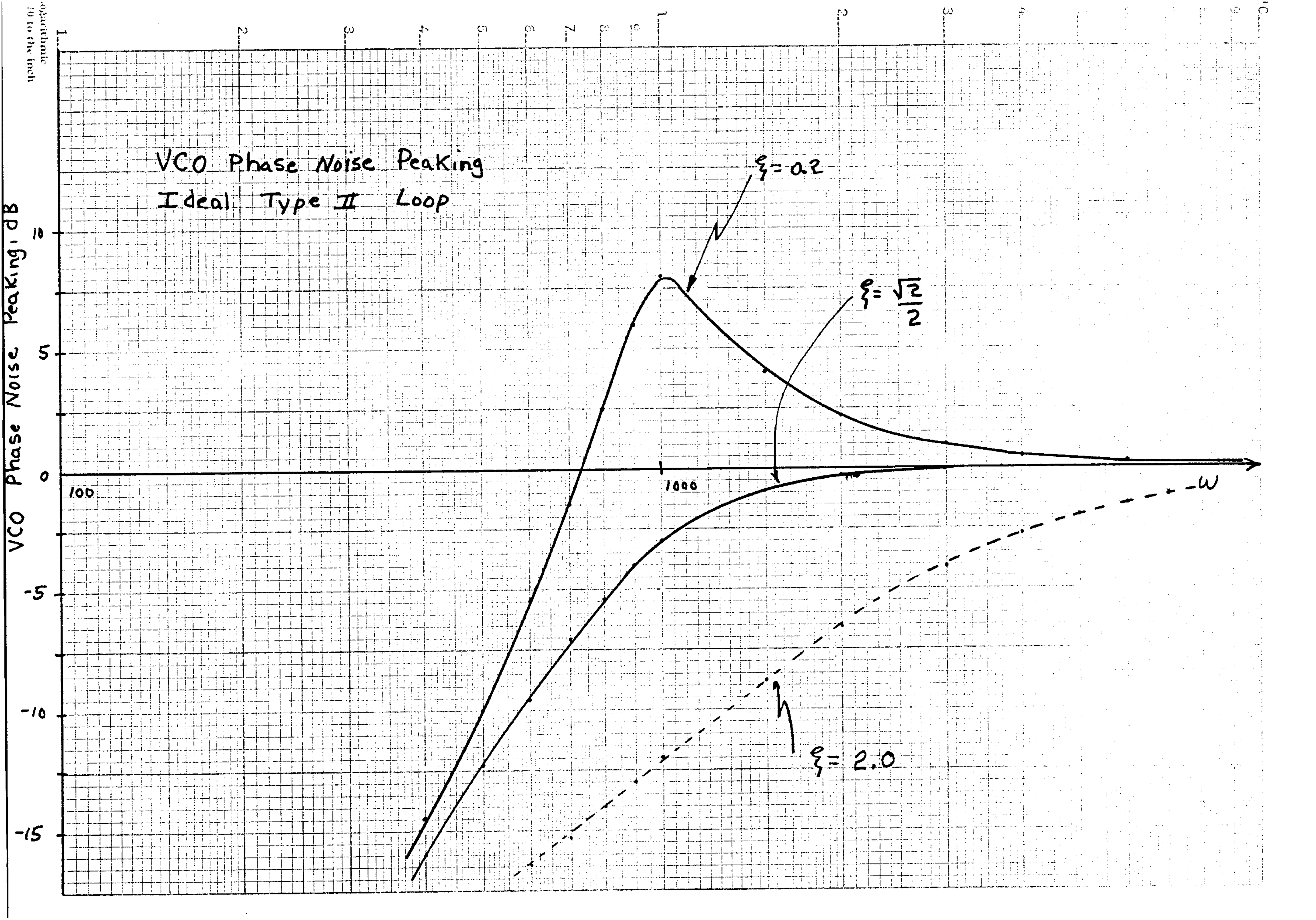


Figure 4.2 Transient phase error $\theta_e(t)$ due to a step in frequency $\Delta\omega$. (Steady-state velocity error, $\Delta\omega/K_v$, neglected.) From Ref. 1 by permission of L. A. Hoffman.

From Gardner [1]


Normalized
3 dB BW
vs.
Damping
Continuous Type II
5
4.5
4
3.5
3
2.5
2
1.5
1
.5
0
0
.2
.4
.6
.8
1
1.2
1.4
1.6
1.8
2
Damping Factor, ξ


PHASE MARGIN vs. DAMPING
TYPE II LOOP
Continuous Analysis
Degrees
100
90
80
70
60
50
40
30
20
10
0
0
.2
.4
.6
.8
1
1.2
1.4
1.6
1.8
2
Damping Factor, ξ


VCO Phase Noise Peaking
Ideal Type II Loop
VCO Phase Noise Peaking, dB
10
5
0
-5
-10
-15
100
1000
ω
ξ = 0.2
ξ = √2/2
ξ = 2.0
Logarithmic
10 to the inch

Phase Noise Transfer Functions

Ideal Type II, I PLL

Loop Type	Divider	VCO
I	$\dfrac{N\,\theta_D(s)}{1+ s/W_N}$	$\dfrac{\theta_V(s)\quad s/W_N}{1+ s/W_N}$
II	$\dfrac{\theta_D(s)\;N\left(1+2s\xi/W_N\right)}{1+2\xi\dfrac{s}{W_N}+\left(\dfrac{s}{W_N}\right)^2}$	$\dfrac{\theta_V(s)\quad (s/W_N)^2}{1+2\xi\dfrac{s}{W_N}+\left(\dfrac{s}{W_N}\right)^2}$

Internal Loop Transport Delay

Open-loop Gain Function $\qquad G_{OL}(s) = \omega_n^2 \dfrac{(1 + 2\xi S/\omega_n)}{S^2} \underbrace{e^{-S\tau_d}}_{\text{Delay Term}}$

Excellent 1st Order Approx $\qquad G_{OL}(s) = \omega_n^2 \dfrac{(1 + 2\xi S/\omega_n)}{S^2} \dfrac{1}{1 + S\tau_d}$

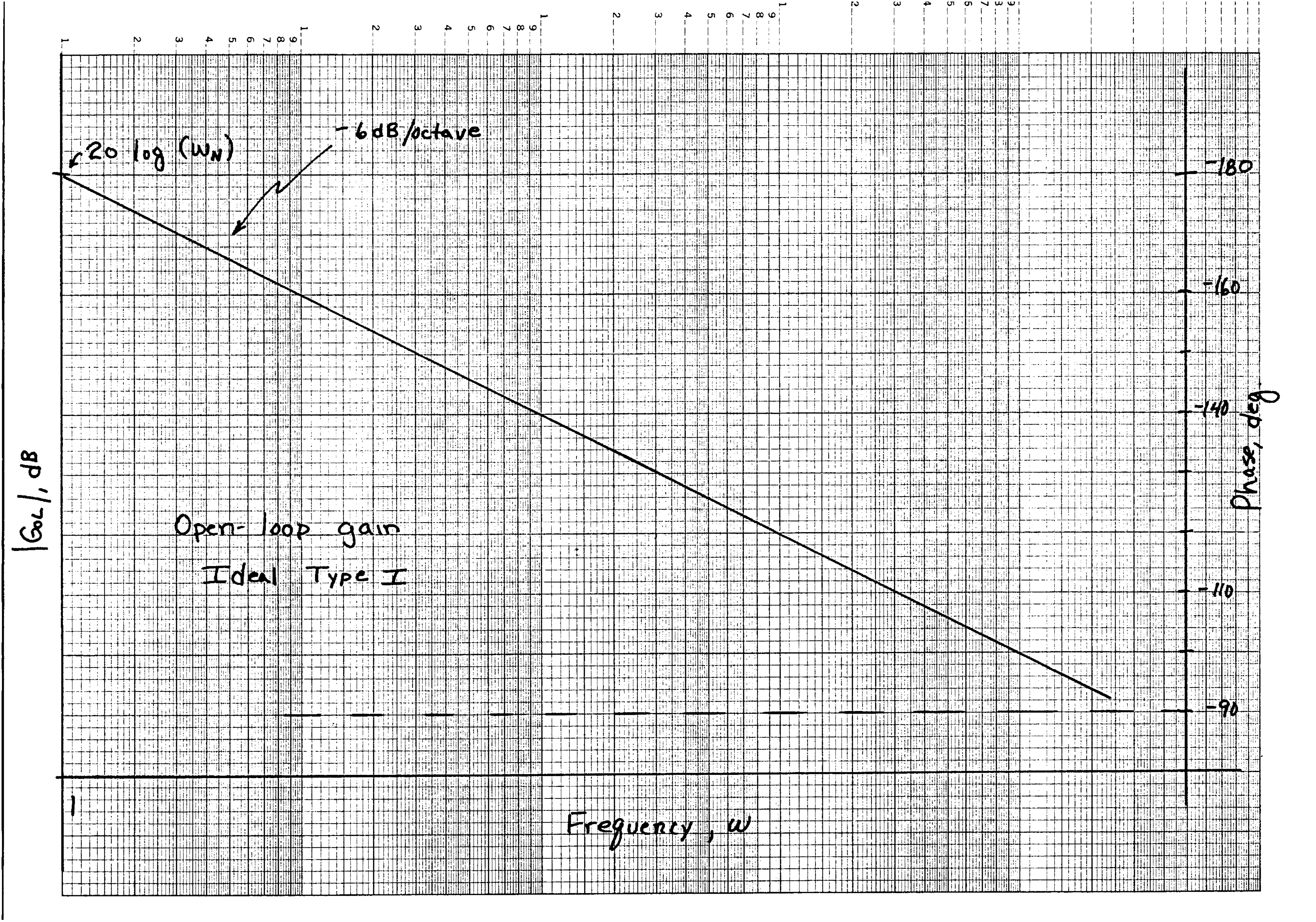
<u>Case:</u> Step Change in Frequency

No closed-form solution possible for exact Type II case.

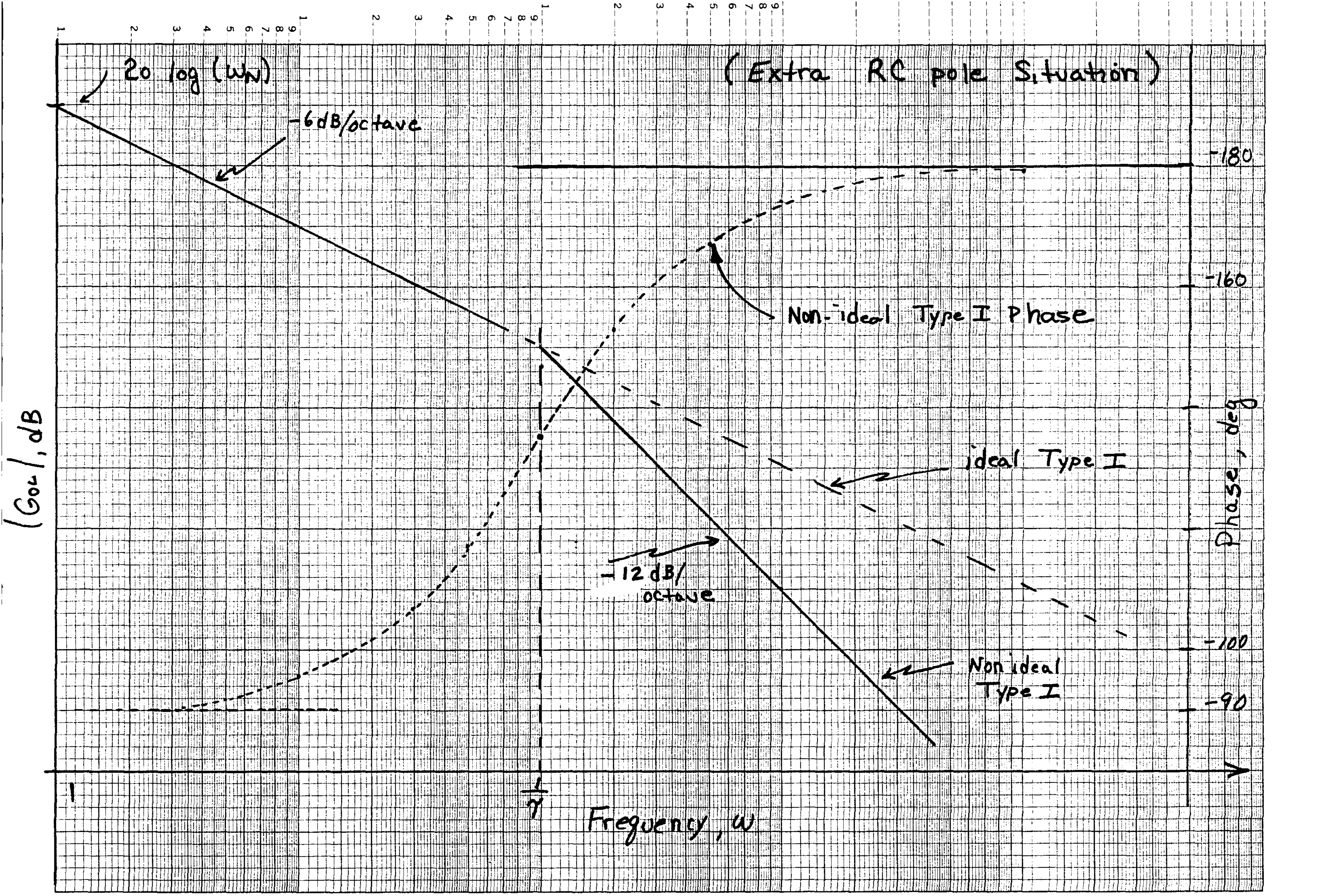
Solution for exact Type I loop with delay:

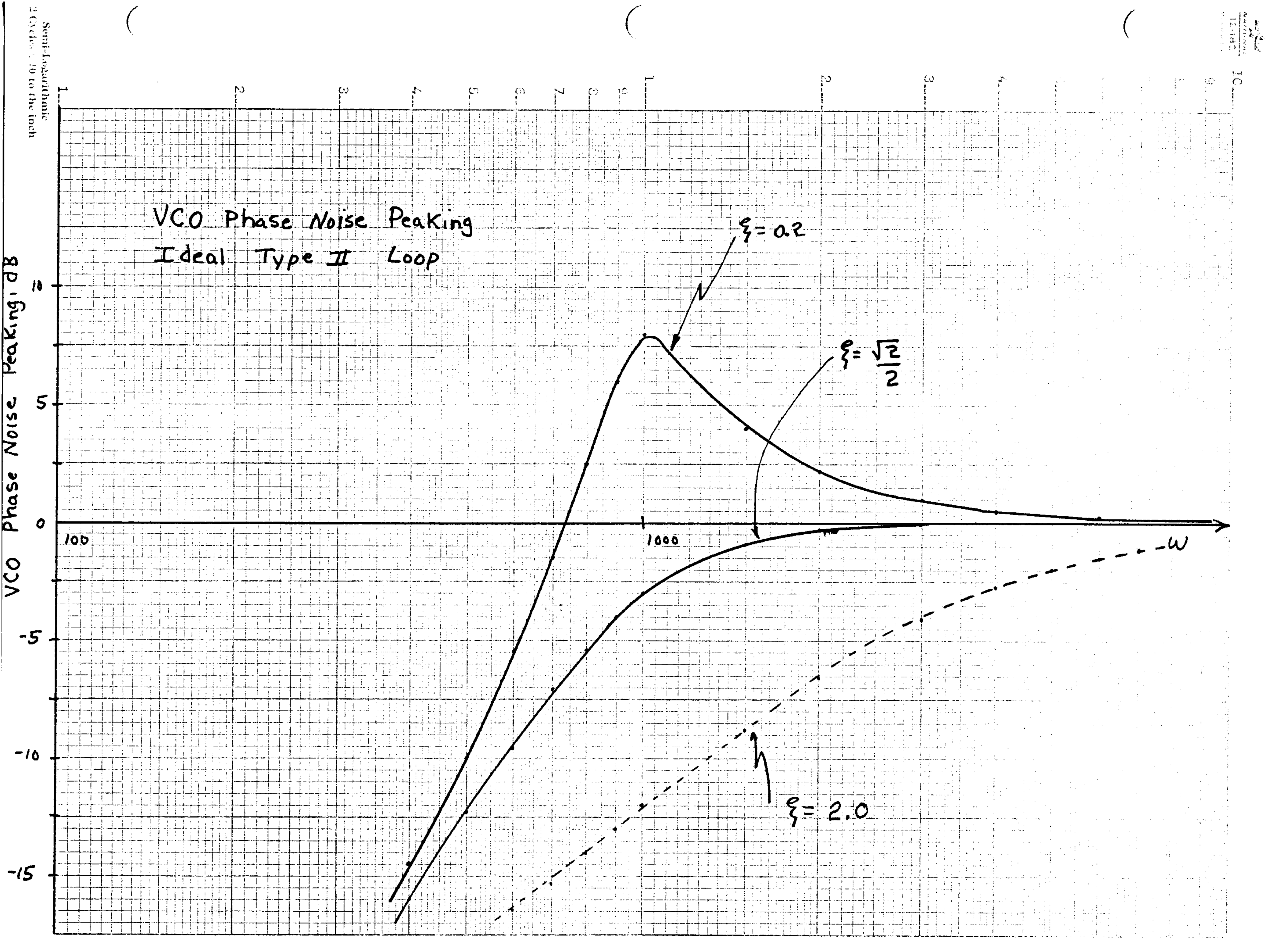
$$\theta_E(t) = \frac{2\pi\,\Delta f}{N}\left[t + \sum_{n=1}^{\frac{t}{\tau}=N} (-\omega_n)^n \frac{(t-n\tau)^{n+1}}{(n+1)!} u(t-n\tau) \right]$$

Time delay is always detrimental to achievable switching speed and phase noise.



20 log (ω_N)
-6dB/octave
$|G_{OL}|$, dB
Open-loop gain
Ideal Type I
Phase, deg.
-180
-160
-140
-110
-90
1
Frequency, ω


20 log (ωN)
(Extra RC pole Situation)
-6dB/octave
-180
-160
Non-ideal Type I Phase
ideal Type I
-12 dB/ octave
Non ideal Type I
-100
-90
|GOL|, dB
Phase, deg
1
1/τ
Frequency, ω



VCO Phase Noise Peaking
Ideal Type II Loop
ξ = 0.2
ξ = √2/2
ξ = 2.0
VCO Phase Noise Peaking, dB
10
5
0
-5
-10
-15
100
1000
ω

Oscillator Phase Noise Models

Leeson [3]

$$S_\phi(f_m) = \frac{FKT}{P_o}\left[1 + \left(\frac{f_o}{2Qf_m}\right)^2\right]$$

Haggai

$$S_\phi(f_m) = \frac{2}{\pi B_c} \frac{1}{1 + \left(\frac{2f_m}{B_c}\right)^2}$$

$$B_c = \frac{\pi}{2} \frac{FKT}{P_o}\left(\frac{f_o}{Q}\right)^2$$

(Equivalent Models)

F	noise factor
T	Absolute Temp
f_o	Center Frequency
Q	Loaded Q
P_o	Power Out
f_m	Offset frequency

Spectral Relationships for Wide-Sense Stationary Processes

$$S_\phi(\omega) = \int_{-\infty}^{\infty} R_\phi(\tau)\, e^{-j\omega\tau}\, d\tau$$

$$R_\phi(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_\phi(\omega)\, e^{j\omega\tau}\, d\omega$$

W. K. Theorem

Variance of Average Frequency Departure

$$\sigma^2\left[\langle\dot{\phi}\rangle_{t,\tau}\right] = \frac{2}{\tau^2}\left[R_\phi(0) - R_\phi(\tau)\right]$$

$$= \frac{2}{\pi\tau^2}\int_{-\infty}^{\infty} S_\phi(\omega)\sin^2\left(\frac{\omega\tau}{2}\right) d\omega$$

Variance of Phase

$$\sigma^2[\phi(t)] = R_\phi(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_\phi(\omega)\, d\omega$$

Variance of Accumulated Phase

$$\sigma^2[\Delta_\tau \phi] = 2\left[R_\phi(0) - R_\phi(\tau)\right]$$

Why Sampled Phase-Locked Loops ?

- Any PLL which includes a divide-by-N in the feedback path is fundamentally a sampled system.

- Exclusion of sampled aspects in large bandwidth PLL's results in considerable error.

- Sampled PLL's are capable of faster switching speed per unit loop-bandwidth than any continuous PLL of the same type.

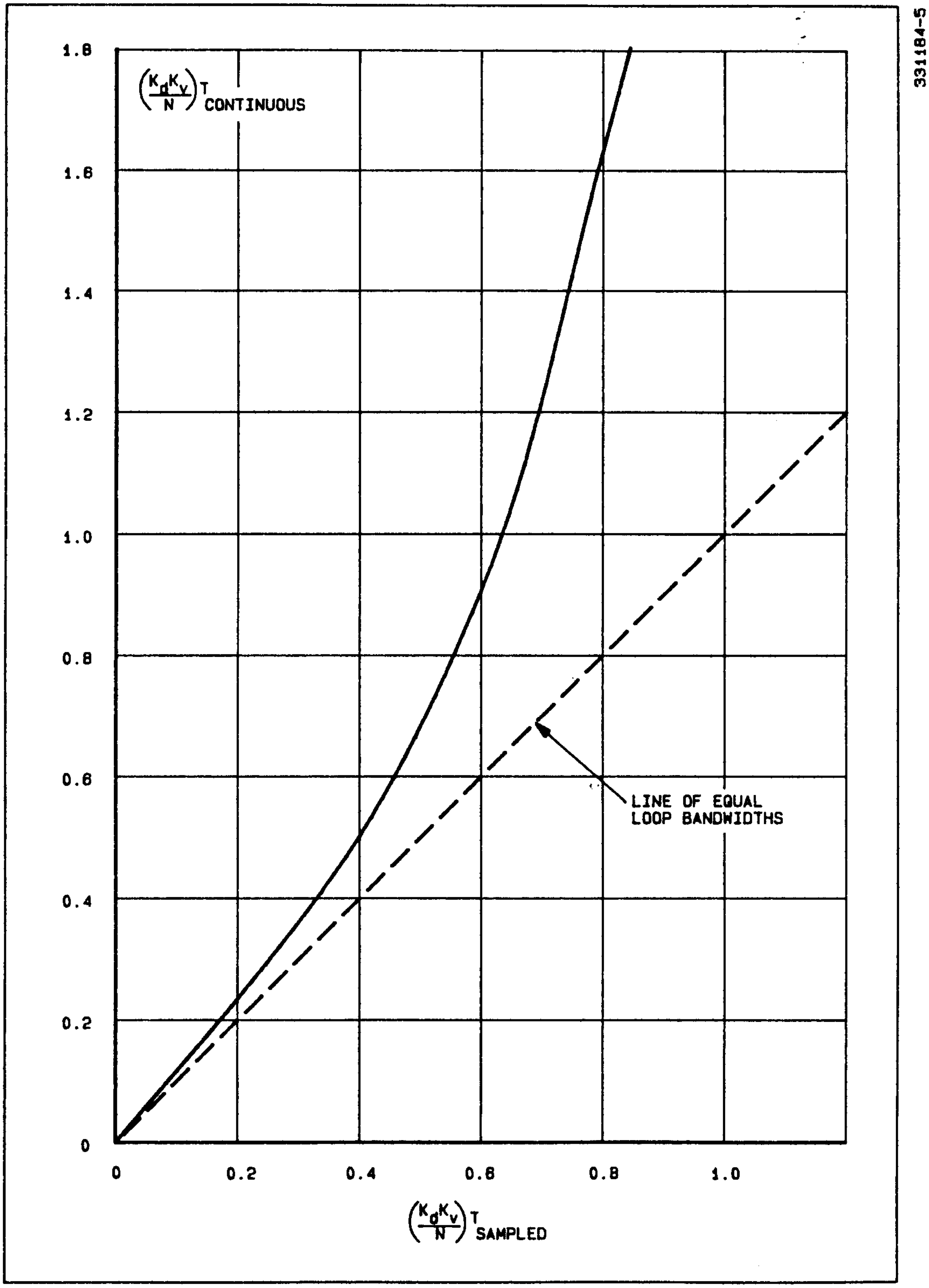


Figure 5. Continuous and Sampled Control System: Phase-Locking Speed [5]

Sampled PLL Model

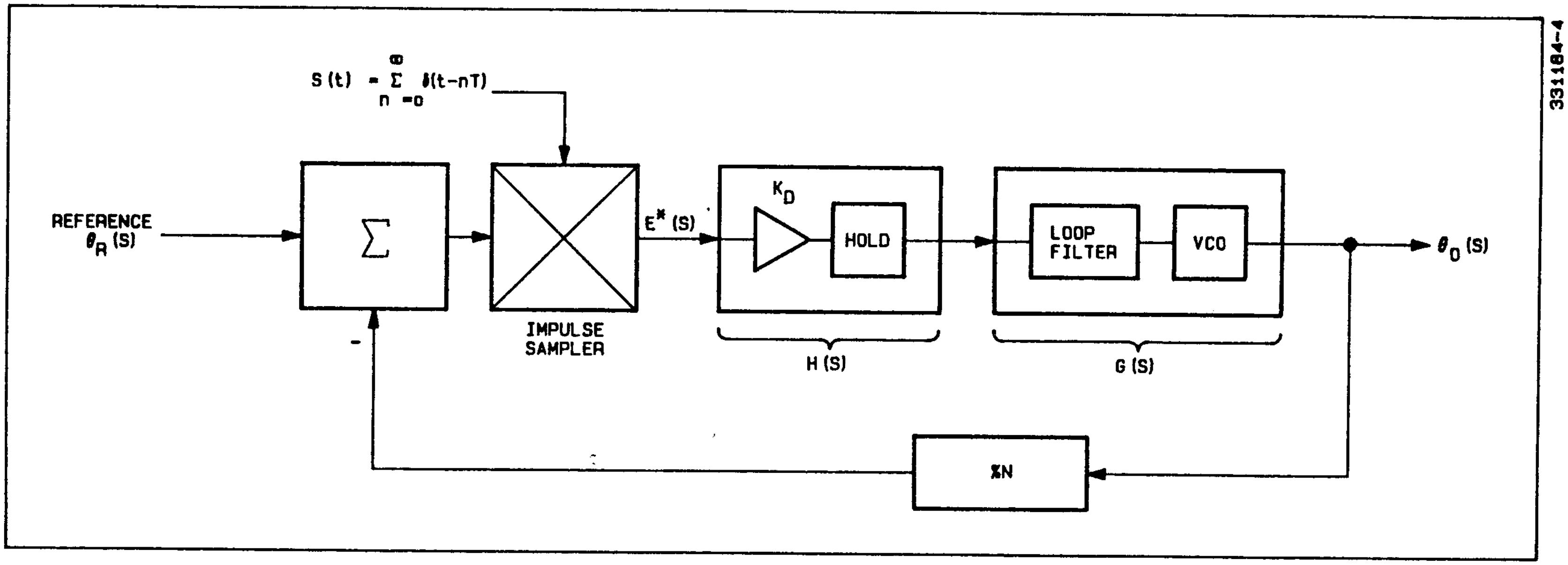
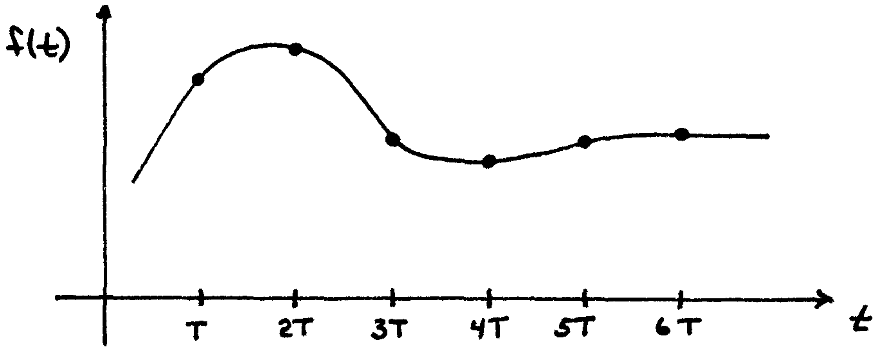
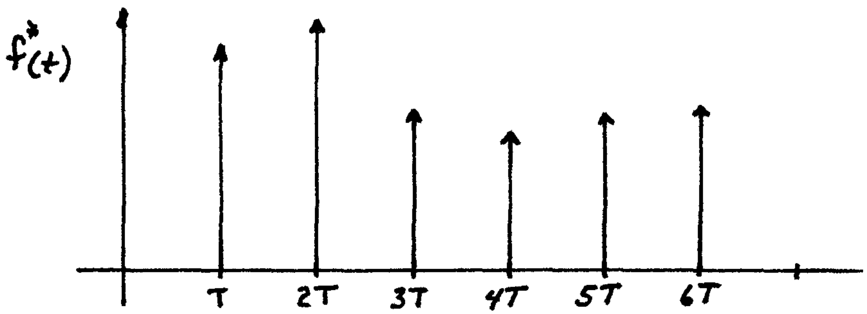


Figure 4. The general Sampled phase-locked loop employs an ideal impulse Sampler which must be followed by some form of ``hold'' device, H(S) [5]

Prelude to Sampled Control Systems





$$\mathcal{L}\left\{f^*(t)\right\} = \sum_{n=0}^{\infty} f(nT)\, e^{-sTn}$$

$$\equiv F(z)$$

Dirac Delta Functions

1) Unit Area $\displaystyle\int_{-\infty}^{\infty} \delta(t)\, dt = 1$

2) Sampling $\displaystyle\int_{-\infty}^{\infty} f(t)\, \delta(t-a)\, dt = f(a)$

Periodic Impulse Sampler

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

$$= \frac{1}{T} \sum_{n=-\infty}^{\infty} e^{jn\omega_s t}$$

$$f^*(t) = f(t)\; p(t)$$

$$= f(t)\; \frac{1}{T} \sum_{n=-\infty}^{\infty} e^{jn\omega_s t}$$

$$\therefore \quad \mathcal{F}\left\{ f^*(t) \right\} = F^*(\omega) = \frac{1}{T} \sum_{n=-\infty}^{\infty} F(\omega - \omega_s n)$$

Sampled PLL Analysis : Error Response [s]

$$E^*(s) = \left[\Theta_R(s) - \frac{\Theta_o(s)}{N} \right]^*$$

$$= \left[\Theta_R(s) - \frac{E^*(s)\ G(s)\ H(s)}{N} \right]^*$$

$$= \Theta_R^*(s) - \frac{E^*(s)\ G^*(s)\ H^*(s)}{N}$$

$$E^*(s) = \frac{\Theta_R^*(s)}{1 + \dfrac{G^*(s)\ H^*(s)}{N}}$$

$$= \frac{\dfrac{1}{T} \displaystyle\sum_{n=-\infty}^{\infty} \Theta_R(s - jn\omega_s)}{1 + \dfrac{1}{NT} \displaystyle\sum_{n=-\infty}^{\infty} G(s+jn\omega_s)\ H(s+jn\omega_s)}$$

Sampled PLL Anaysis : Z-Transforms [5]

Type I

$$G_{OL}(z) = \frac{K}{z-1} \;; \qquad K = \frac{K_d\, K_V}{N}\, T \qquad (\textit{Ideal Loop})$$

$$\theta_e(nT) = \frac{2\pi\, \Delta f}{K_d\, K_V}\left[\, 1 - (1-K)^n \,\right] \qquad \text{Radians}$$

$$\text{Gain Margin} = -20 \log\left(\pi \frac{\omega_N}{\omega_S}\right) \;; \qquad \omega_N = \frac{K_d\, K_V}{N}$$

For $K \equiv 1$, Gain Margin = 6 db

(Exclusion of high order components ré aliasing would predict gain margin 13.1 db)

Sampled PLL Analysis [5]

Type II

$$G_{OL}(z) = \frac{K_d \; K_v \, T \left\{ \left[T/2 + \tau_2 \right] z + \left[T/2 - \tau_2 \right] \right\}}{N \; \tau_1 \; (z-1)^2}$$

$$\theta_e(nT) = \frac{2\pi \, \Delta f \, T}{a-b} \left[a^n - b^n \right]$$

$$a, b = \frac{-A \pm \sqrt{A^2 - 4B}}{2}$$

$$A = 2 - K \, \frac{T/2 + \tau_2}{\tau_1}$$

$$B = 1 + K \, \frac{T/2 - \tau_2}{\tau_1}$$

$$K = \frac{K_d \, K_v}{N} \, T$$

$$\text{Gain Margin} = -20 \log \left[2\pi \zeta \, \frac{\omega_N}{\omega_s} \right] \qquad \omega_N^2 = \frac{K_d \, K_v}{N \tau_1}$$

$= 2.5$ db for speed-optimized PLL

Sampled PLL Anaysis [6]

Type II with Internal Time Delay

$$G_{OL}(z) = \frac{\omega_N^2 \, (a z^2 + bz + c)}{(z-1)^3}$$

$$\theta_e(z) = \frac{\Delta\theta \, z^3 + (2\pi \Delta f T - \Delta\theta) \, z^2}{z^3 + z^2(a\omega_n^2 - 2) + z(1 + b\omega_n^2) + \omega_n^2 C}$$

$$\omega_n^2 = \frac{K_d K_v}{N \tau_1}$$

$\Delta\theta$: Phase error step, Rad.
Δf = frequency step, Hz

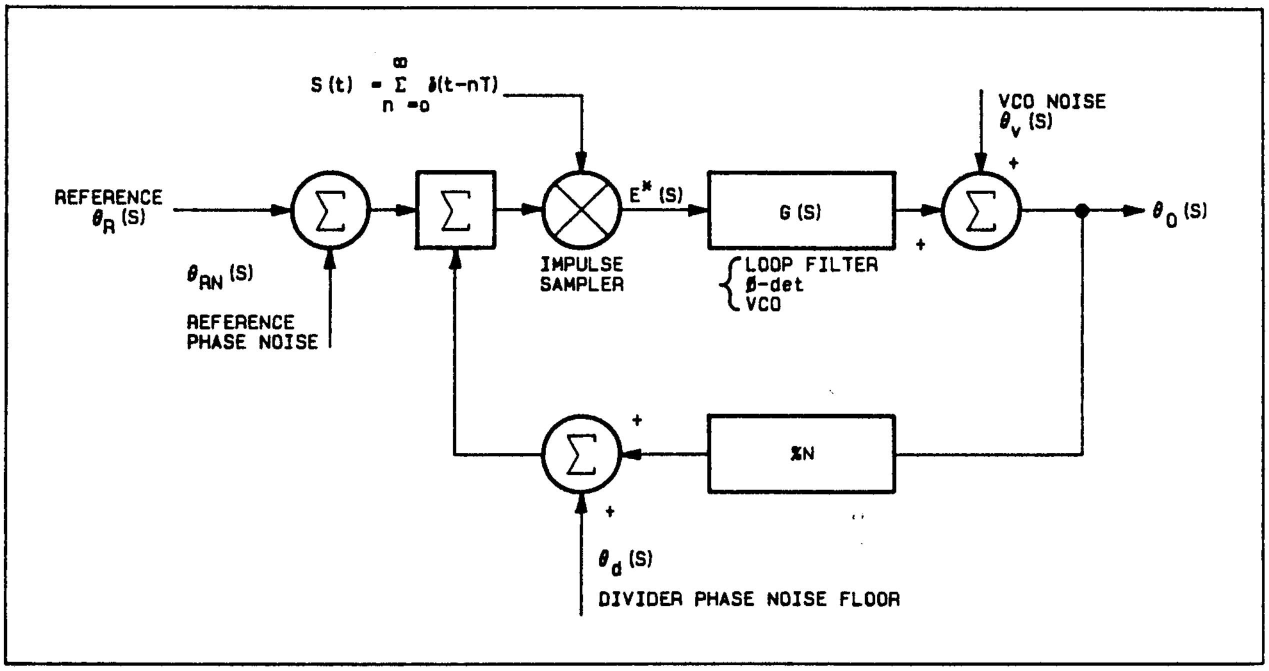
$$m = 1 - \tau_{delay}/T$$

$$a = \frac{(mT)^2}{2} + mT\tau_2$$

$$b = \frac{T^2}{2}(2m - 2m^2 + 1) - 2m\tau_2 T + \tau_2 T$$

$$c = \frac{T^2}{2}(m-1)^2 + \tau_2 (m-1) T$$

Sampled PLLs: Phase Noise Considerations [5]



Phase Noise [5]

Reference noise effect upon the output phase noise:

$$\theta_o(s)\Big|_{Ref} = \frac{\theta^*_{RN}(s)\quad G(s)}{1 + \dfrac{G^*(s)}{N}}$$

Divider noise effect upon the output phase noise:

$$\theta_o(s)\Big|_{Div} = \frac{\theta^*_d(s)\quad G(s)}{1 + \dfrac{G^*(s)}{N}}$$

VCO noise effect upon the output phase noise spectrum:

$$\theta_o(s)\Big|_{VCO} = \frac{\theta_V(s) + \left[\theta_V(s)\,G^*(s) - \theta^*_V(s)\,G(s)\right]/N}{1 + \dfrac{G^*(s)}{N}}$$

Not necessarily 0

$$\approx \frac{\theta_V(s)}{1 + \dfrac{G^*(s)}{N}}$$

Simplified Phase Noise : (Type I Loop Only, Ideal) [5]

Reference Noise

$$\frac{S_o(\omega)}{S_{RN}(\omega)} = \frac{16\, K^2 N^2 \sin^4(\omega T/2)}{(\omega T)^4 \left(K^2 + 2 - 2K + 2(K-1)\cos(\omega T)\right)} \quad ; K = \frac{K_d K_v T}{N}$$

$$= N^2 \left[\frac{\sin(\omega T/2)}{(\omega T/2)} \right]^4 \qquad \text{for } K \equiv 1$$

Divider Noise

Same as reference case

VCO Noise

$$\frac{S_o(\omega)}{S_v(\omega)} = \frac{4 \sin^2(\omega T/2)}{K^2 + 2 - 2K + 2(K-1)\cos(\omega T)}$$

$$= 4 \sin^2\left(\frac{\omega T}{2}\right) \qquad \text{for } K \equiv 1$$

Practical Considerations

* Power Supply Filtering
* Post-Tuning Drift
* Feedback Dividers
* Phase Noise Considerations
* Non-ideal Features
  - VCO linearity
  - Sampling Efficiency
  - Transport Delay

References

1. Gardner F.M. Phaselock Techniques

2. Viterbi A.J. Principles of Coherent Communications

3. Leeson D.B. "A Simple Model of Feedback Oscillator Noise Spectrum" *IEEE Proc.* Feb 1966

4. Cutler L.S. "Some Aspects of the Theory and Measurement of Frequency Fluctuations in Frequency Standards" *IEEE Proc.* Feb. 1966

5. Crawford J.A. "Understanding the Specifics of Sampling in Synthesis" Microwaves & RF Aug., Sept. 1984

6. Crawford J.A. "Frequency Synthesizer IR&D" *TRW IOC C423.25-85* May 23, 1985

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7. Ward C.J. "Delay Reduction Techniques in Phase-Locked Loop Amplifiers" *Electronic Letters* Apr. 2, 1981

8. Crawford J.A. "Frequency Synthesizer IR&D" TRW IOC C423.25-91 June 5, 1985

9. Crawford J.A. "Universal L/X Band Frequency Synthesizer" TRW IOC C423.25-94 July 9, 1985

10. Baugh R.A. "Low Noise Frequency Multiplication" Frequency Control Symposium

11. Andresen S.G. "Phase Noise of Various Frequency Doublers" IEEE IM-22 June, 1973

12. Egan W.F. "The Effects of Small Contaminating Signals in Nonlinear Elements Used in Frequency Synthesis & Conversion" IEEE Proc. July, 1981

13. Rotholz E. "Phase Noise of Mixers" Electronic Letters 13 Sept. 1984

14. Kroupa V.F. "Noise Properties of PLL Systems" IEEE COM-30 Oct. 1982

15. Walls F.L. et al "Design Considerations in State-of-the-Art Signal Processing and Phase Noise Measurement Systems" Frequency Control Symposium 1976

16. Peregrino L "Phase Noise Measurement Using a High Resolution Counter with On-Line Data Processing" Frequency Control Symposium

17. Noordanus J. et al "Direct-division Phase-Lock Loop at 12 GHz IEE Proc. Dec. 1983

18. Goldwasser W.J. "Design Shortcuts for Microwave Frequency Dividers" Electronic Engineer May, 1970

19. Upadhyayula L.C. "Microwave Frequency Dividers" RCA Review Dec. 1973

20. Harrison R.G. "A Broad-Band Frequency Divider Using Microwave Varactors" IEEE MTT-25, Dec. 1977